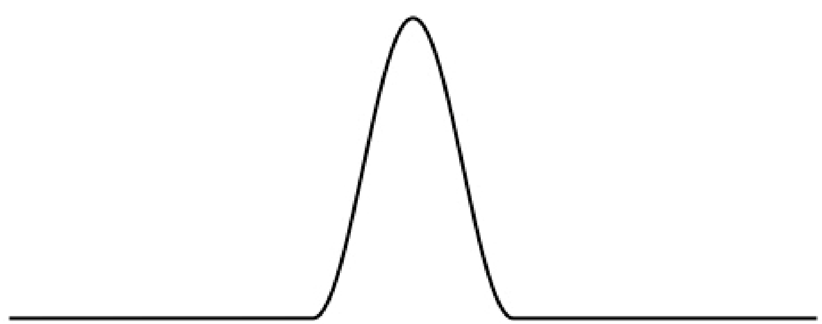
Phase Mixing of Partially Confined Alfvén Waves

Alex Prokopyszn, Alan Hood, Ineke De Moortel

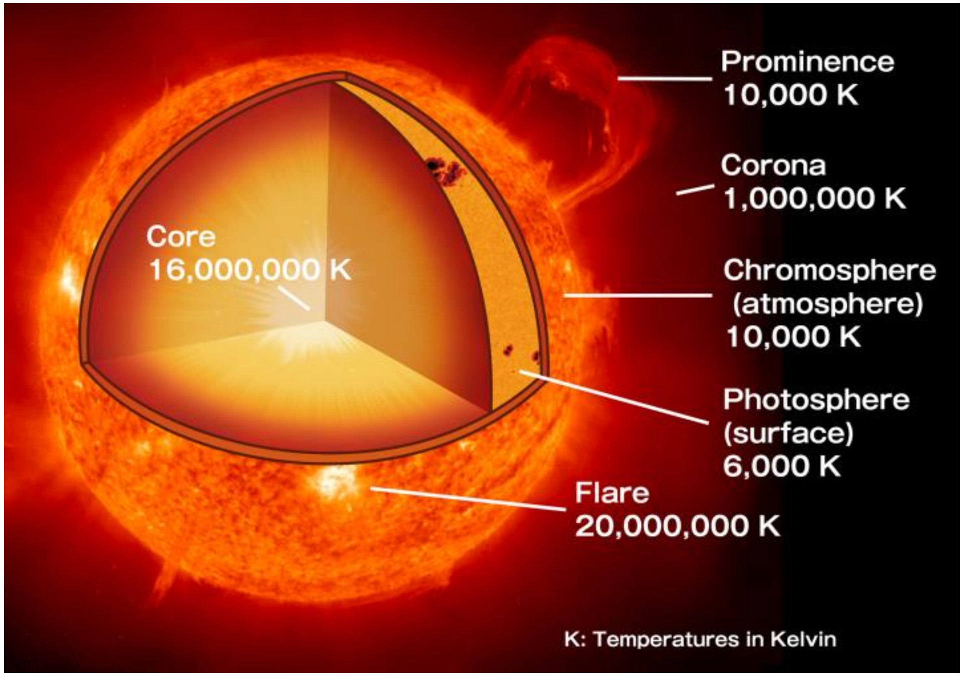


Aims

- Study continuous, footpoint driven Alfvén waves in coronal-like plasma.
- Analyse wave leakage through the transition region.
- Show that an upper limit for the gradients is reached.
- Calculate and present this upper limit.

Background

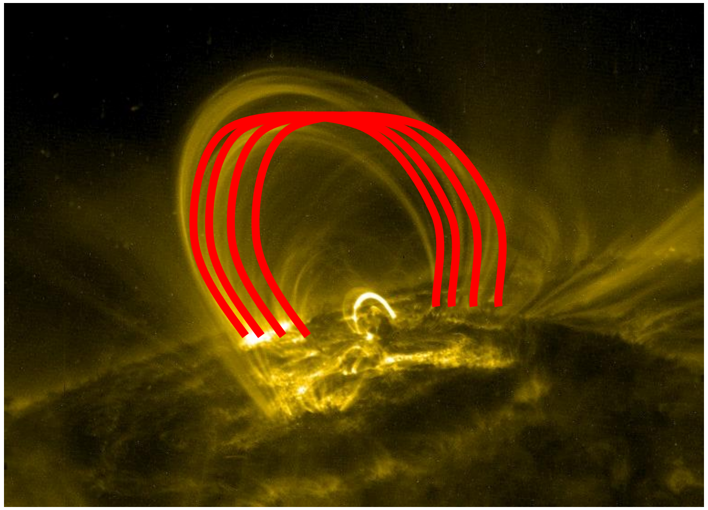
- Many proposed mechanisms for coronal heating, e.g. turbulence, phase mixing, nanoflares etc..
- It is difficult to show which mechanisms are dominant (if any).



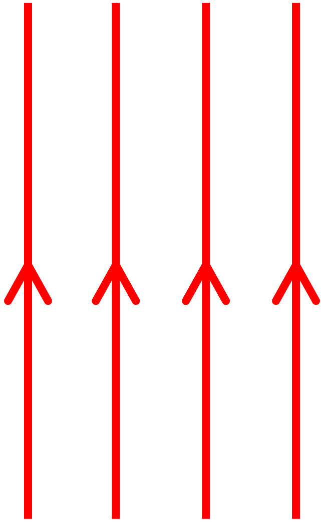
Source: aasnova.org

Background

- Can we quantify an upper bound for the heating provided by phase mixing?
- Estimate a parameter space where phase mixing is negligible.
- Upper bound is easier to find than an accurate value.



$\approx$



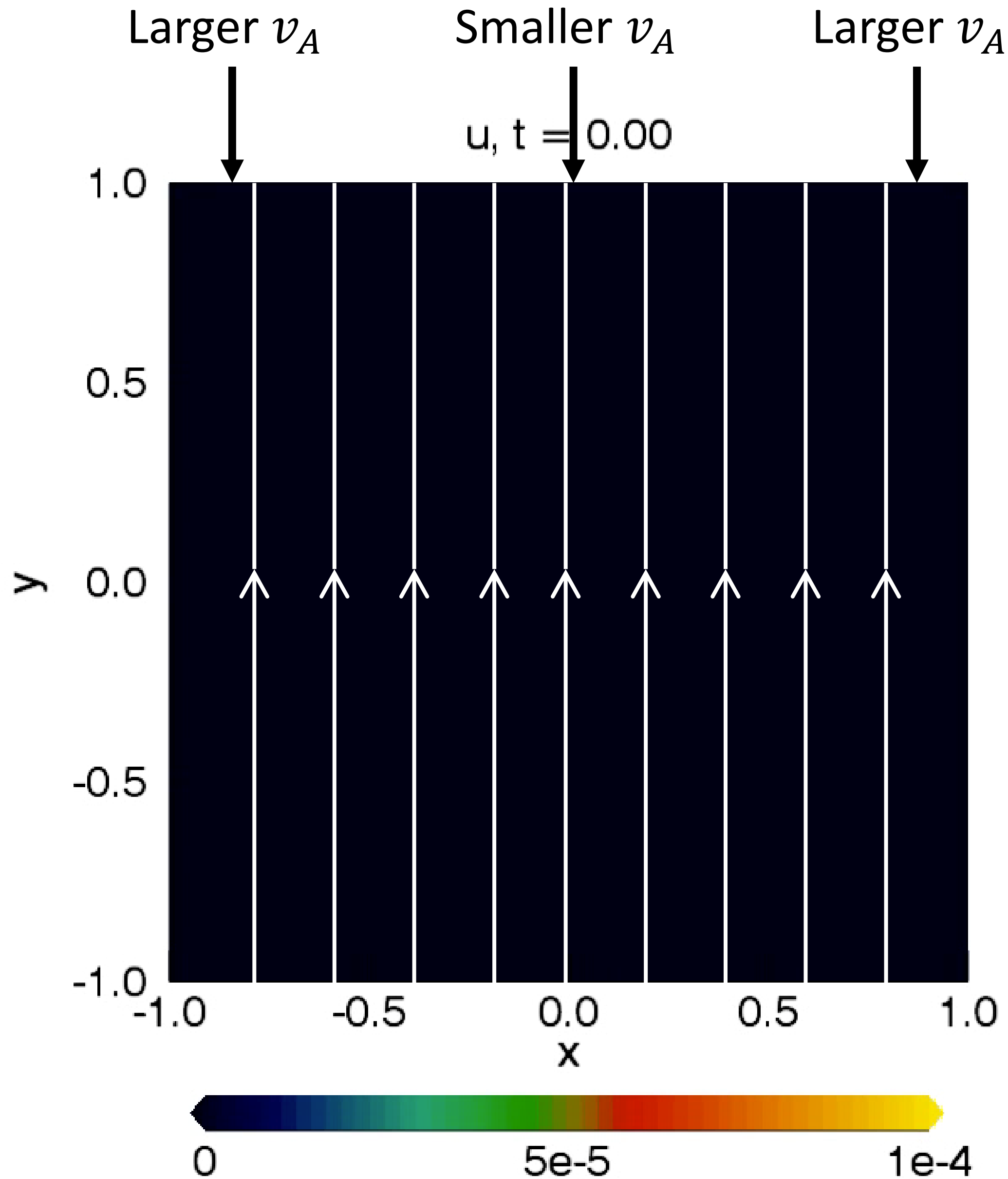
Source: TRACE, 171 Å,
Characterstic temperature $\approx 6 \times 10^5 \mathrm{K}$

Phase mixing

$$\underbrace{\nabla_\perp v_A \Longrightarrow \uparrow \nabla_\perp u, \nabla_\perp b}_{\text{This process = phase mixing.}}$$

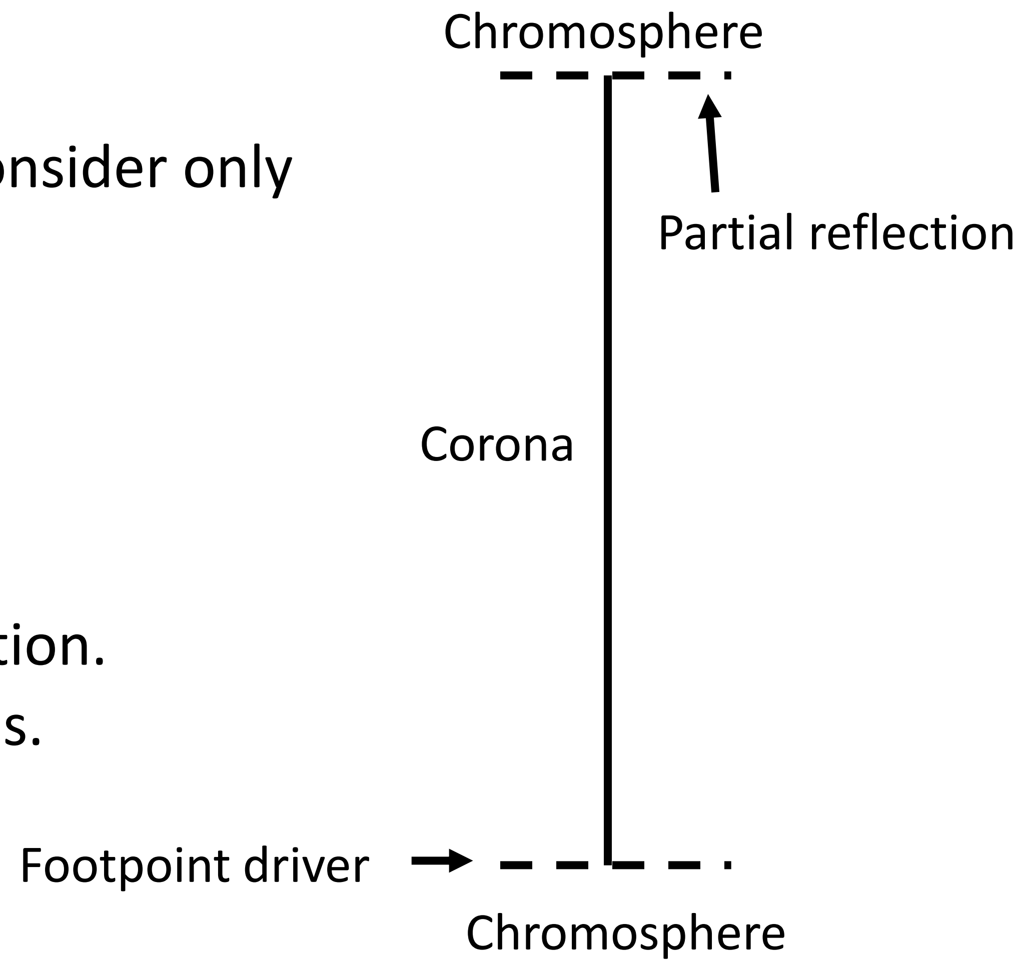
- In ideal MHD phase mixing is only responsible for changes to $\nabla_\perp$ not $\nabla_{||}$.
- Ohmic heating rate of an Alfvén wave is given by

$$|\nabla b|^2/\sigma.$$



Our model of a coronal loop

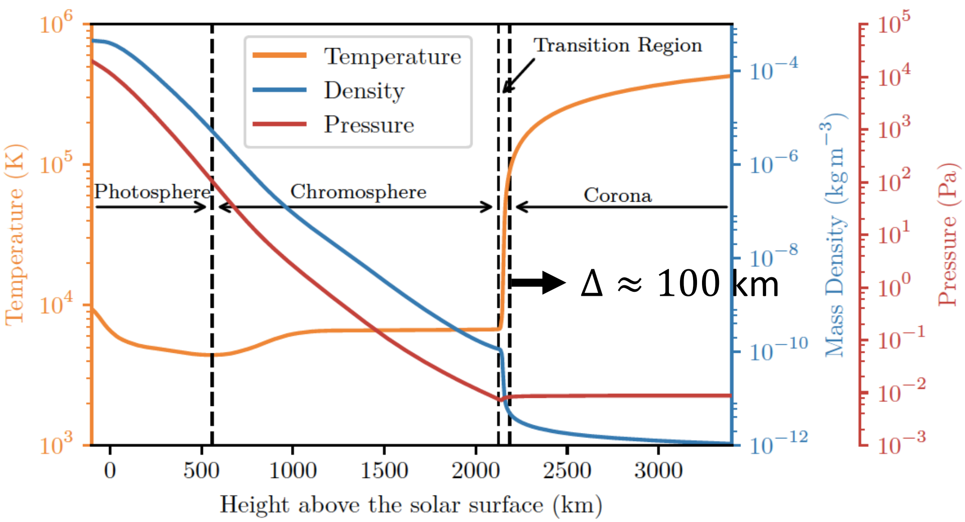
- Model loop in 2.5D, but consider only one field line.
- $\nabla_{||} v_A = 0^*$.
- $\nabla_{\perp} v_A \neq 0$.
- Ideal MHD.
- Linear waves.
- There is an invariant direction.
- Consider only Alfvén waves.

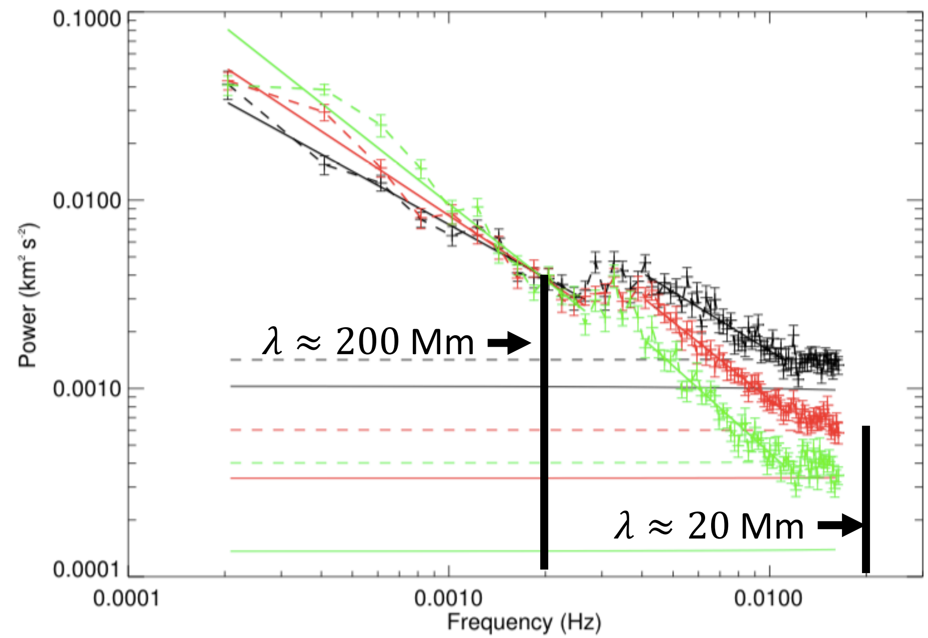


Structure

- Estimate wave reflection/transmission coefficient.
- Experiment 1:
  - Full reflection, sinusoidal driver.
- Experiment 2:
  - Partial reflection, sinusoidal driver.
- Experiment 3:
  - Partial reflection, broadband driver.
- Conclusions

The transition region is approximately a discontinuity in comparison to the wavelength of observed waves



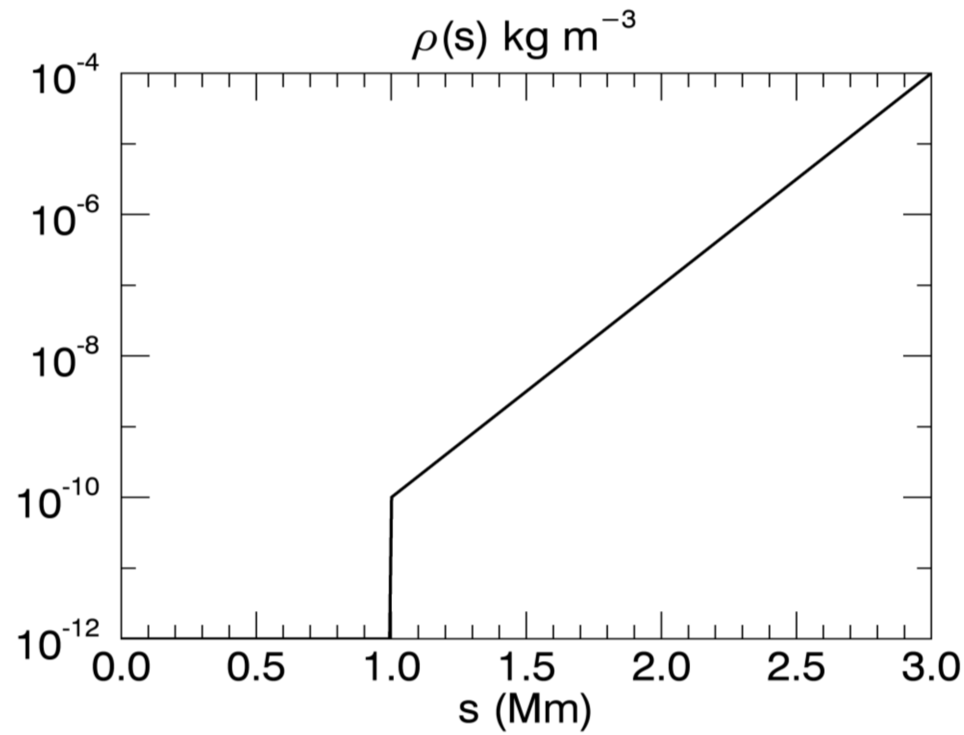


- Plasma values given by VAL model of the solar atmosphere.

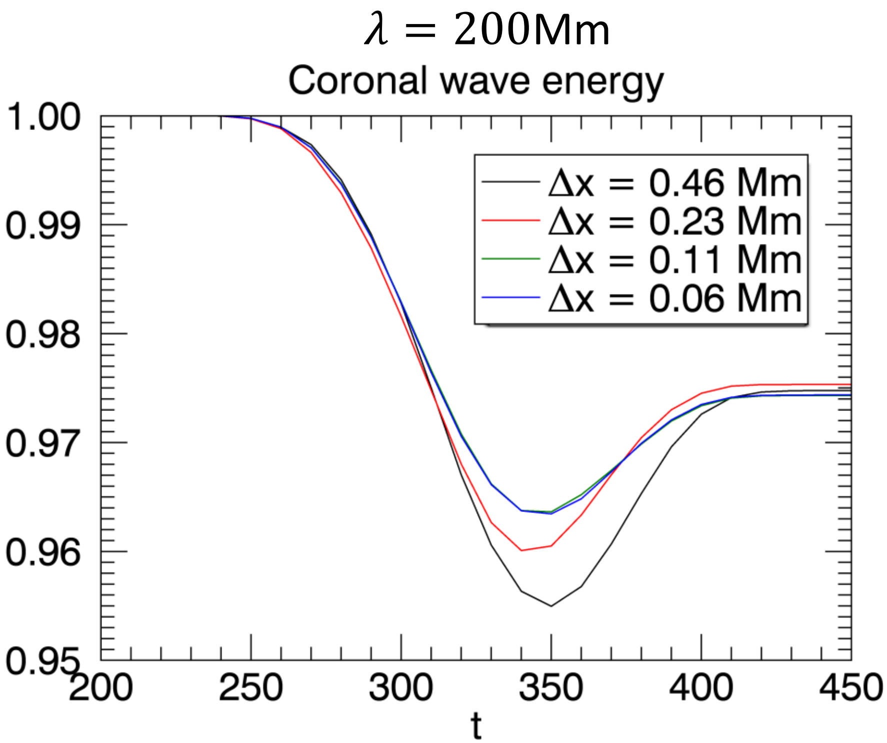
- Power spectrum from Morton et al. (2016).
- Used COMP
- Cadence ≈ 0.03 Hz

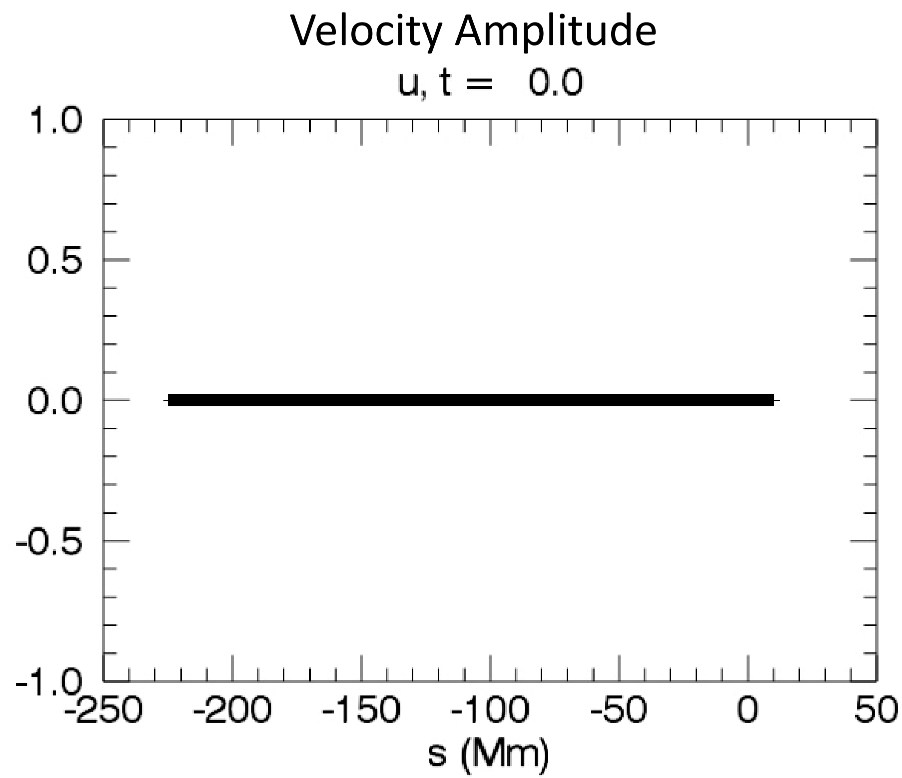
Estimating the energy reflection coefficient (R_E)

- Use LARE2D (but only 1 cell wide).
- Use density values based on VAL.
- Send in an Alfven pulse and measure how much energy reflects and transmits.



Calculating the energy reflection coefficient (R_E)



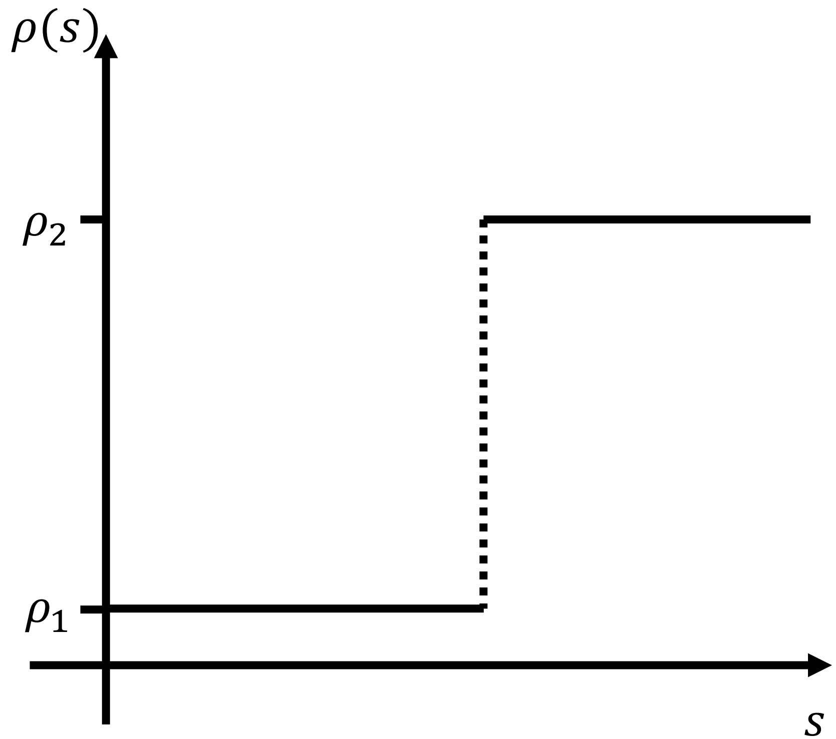


Energy reflection coefficient (R_E)

Grid size (Δs) (Mm)	R_E ($\lambda = 20$ Mm) ($P \approx 50$ s)	Grid size (Δs) (Mm)	R_E ($\lambda = 200$ Mm) ($P \approx 500$ s)
$\approx 5.9 \times 10^{-2}$	≈ 0.843	$\approx 4.6 \times 10^{-1}$	≈ 0.975
$\approx 2.9 \times 10^{-2}$	≈ 0.838	$\approx 2.3 \times 10^{-1}$	≈ 0.975
$\approx 1.5 \times 10^{-2}$	≈ 0.841	$\approx 1.1 \times 10^{-1}$	≈ 0.974
$\approx 0.7 \times 10^{-2}$	≈ 0.839	$\approx 0.6 \times 10^{-1}$	≈ 0.974

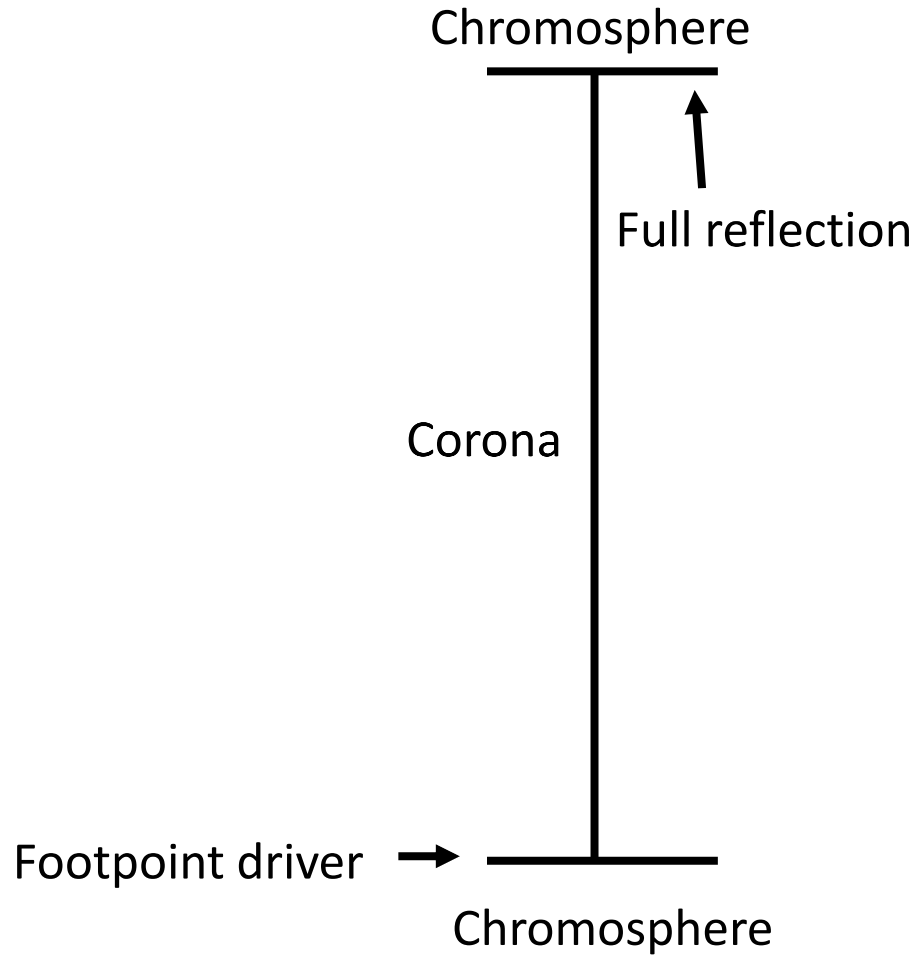
Analytic check of R_E

- R_E of the configuration on the right is
$$R_E = \left(\frac{\sqrt{\rho_2}-\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}}\right)^2.$$
- For $\frac{\rho_2}{\rho_1} = 100$ this gives
$$R_E \approx 0.77.$$
- This is less than R_E for numerical experiments as expected.
- For $\frac{\rho_2}{\rho_1} = 1000$ this gives
$$R_E \approx 0.88.$$



Experiment 1/3

- Full reflection ($R = 1$).
- Sinusoidal driver.



Analytic solution

- Solve the wave equation.

$$\frac{\partial^2 u}{\partial t^2} = v_A^2 \frac{\partial^2 u}{\partial s^2}$$

- Domain: $-l \leq s \leq l$
- Conditions:
  - Driver: $u(-l, t) = \sin(\omega t)$
  - Require a factor $R \leq 1$ to reflect at $s = \pm l$.
- Solve using d'Alembert's formula.

Analytic solution

Heaviside step function

$$u = \sum_{k=0}^{m} (-1)^k R^k H(\theta_k) \sin(\omega \theta_k),$$

$$\theta_k = t - (-1)^k \frac{s}{v_A} - \frac{2k+1}{v_A} l,$$

$$m = \left\lfloor \frac{t v_A}{L} \right\rfloor.$$

Floor function

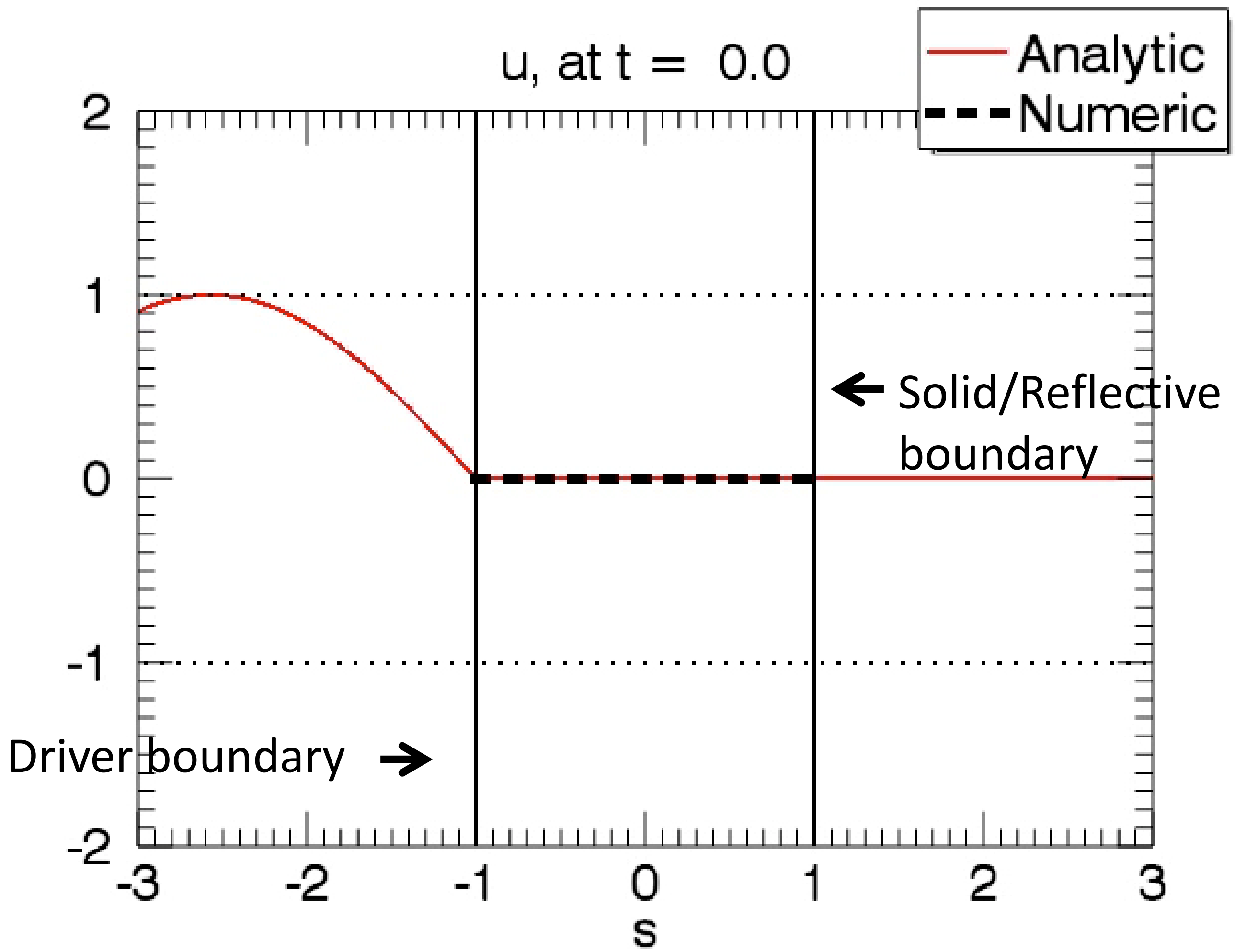
Numerical solution

- Obtained using LARE2D.
- Linear driver imposed on the left boundary of the form:

$$u = 10^{-4} \sin(\omega t) \, .$$

- Solid boundary conditions otherwise.

Numeric vs Analytic
Non-resonant driver



Solution for a resonant driver

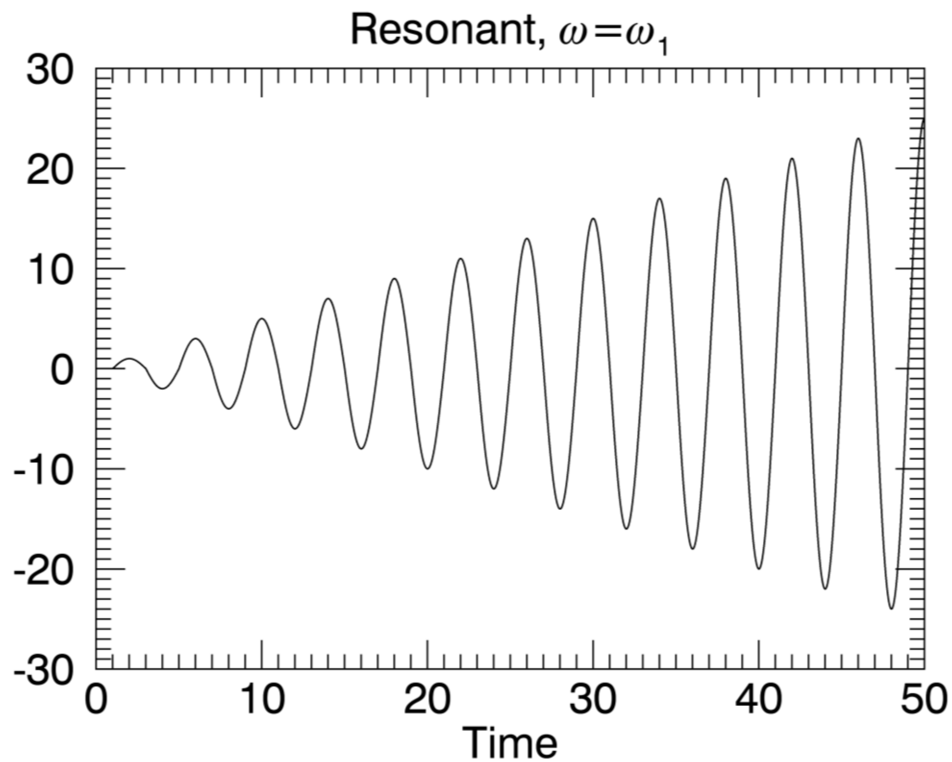
Amplitude, u, at loop apex

- The natural frequencies are given by,

$$\omega_n = n\frac{v_A}{L}\,\pi.$$

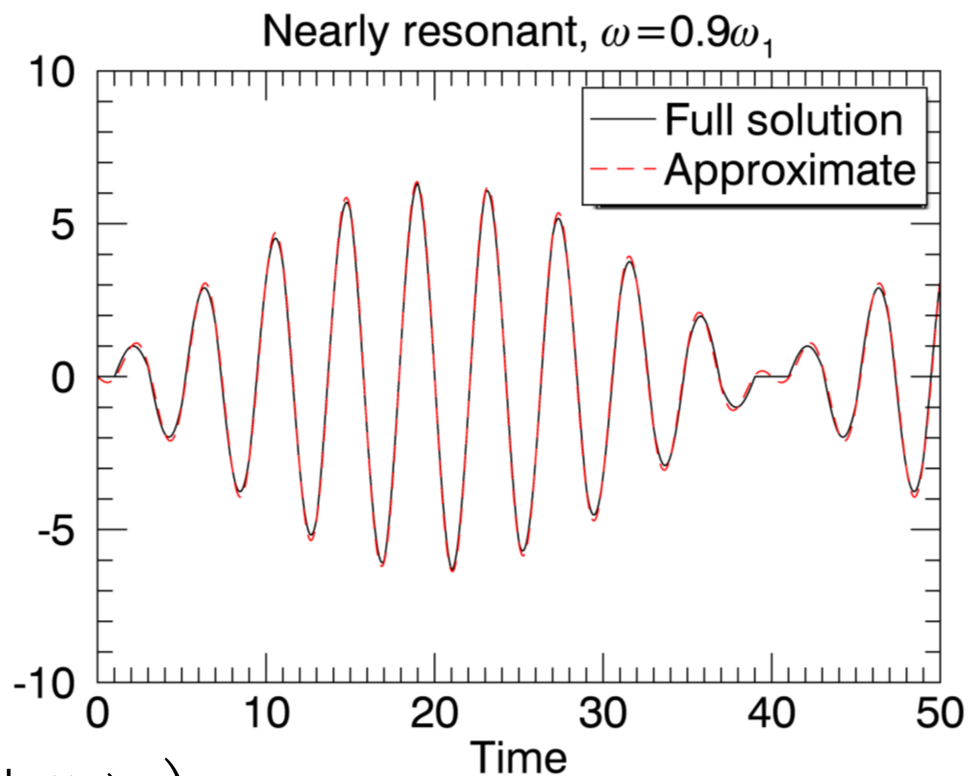
- If $\omega = \omega_n, n =$ odd integer, then the solution is

$$u \approx \pm t\frac{v_A}{L}\cos(\omega \mathrm{t}).$$



Solution near resonance

Amplitude, u, at loop apex



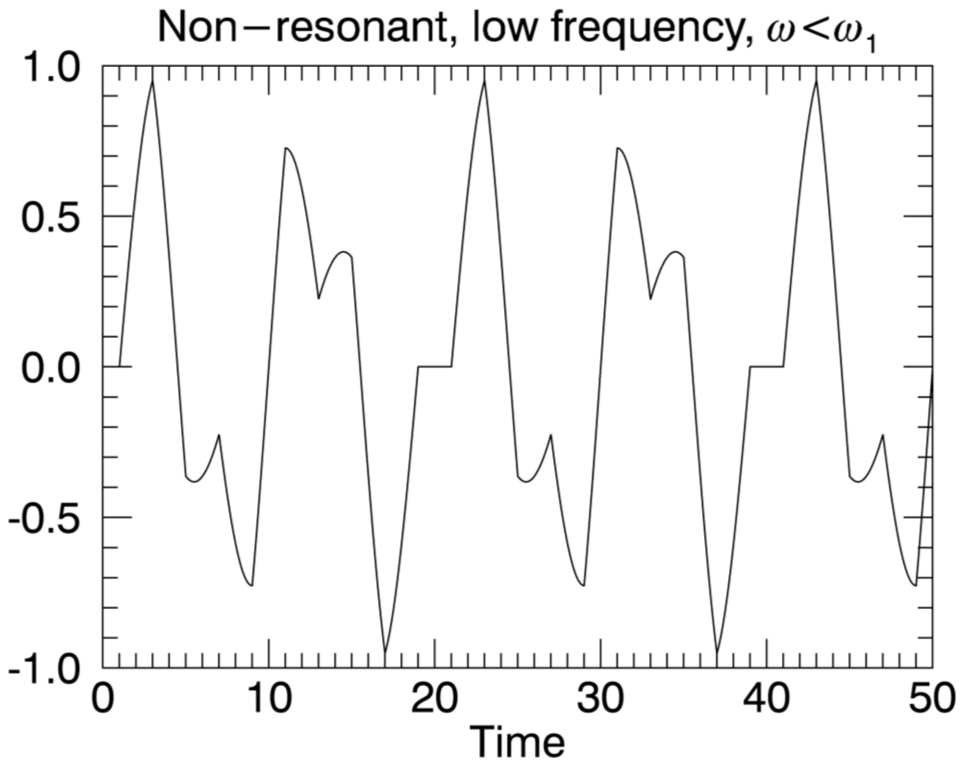
- If $\omega \approx \omega_n, n =$ odd integer then the solution is,

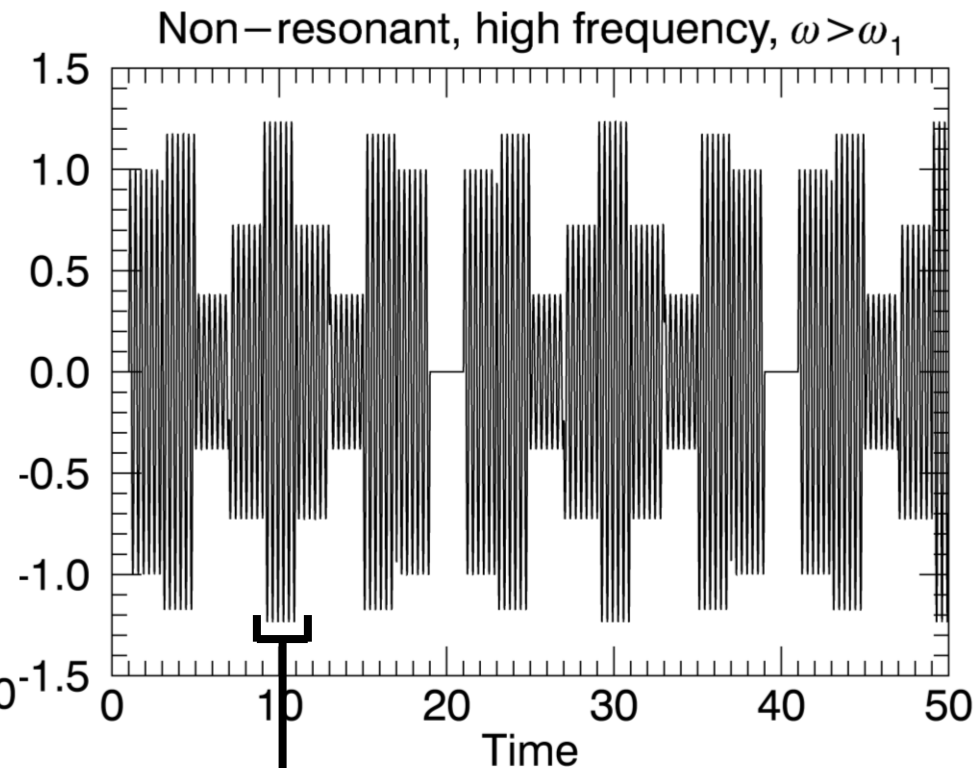
$$u \approx \sec\left(\frac{\omega l}{v_A}\right) \sin\left(\left(\frac{\omega - \omega_n}{2}\right) t\right) \cos\left(\left(\frac{\omega + \omega_n}{2}\right) t\right).$$

Beating frequency.

Solution away from resonance

Amplitude, u, at loop apex

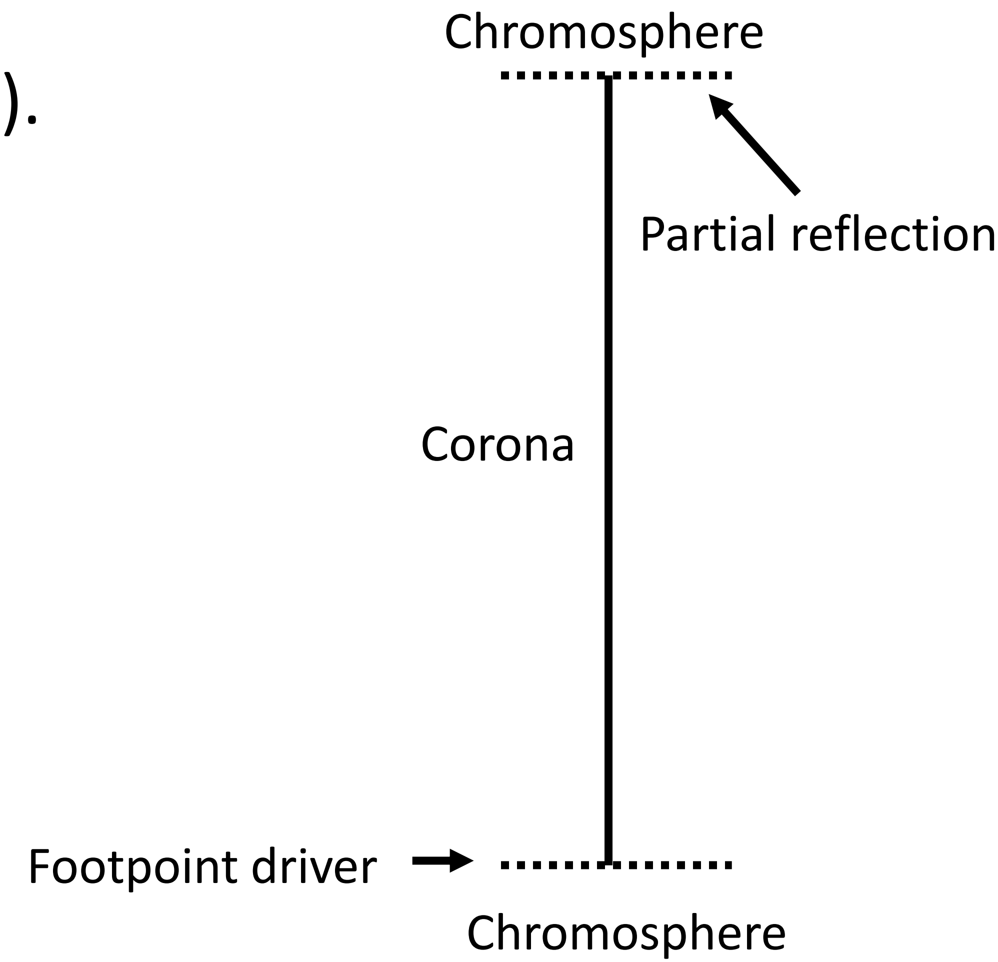




$$\Delta \mathrm{t} = \frac{L}{v_A}$$

Experiment 2/3

- Parital reflection ($R < 1$).
- Sinusoidal driver.



Analytic solution

- Same as before, except now $R < 1$.

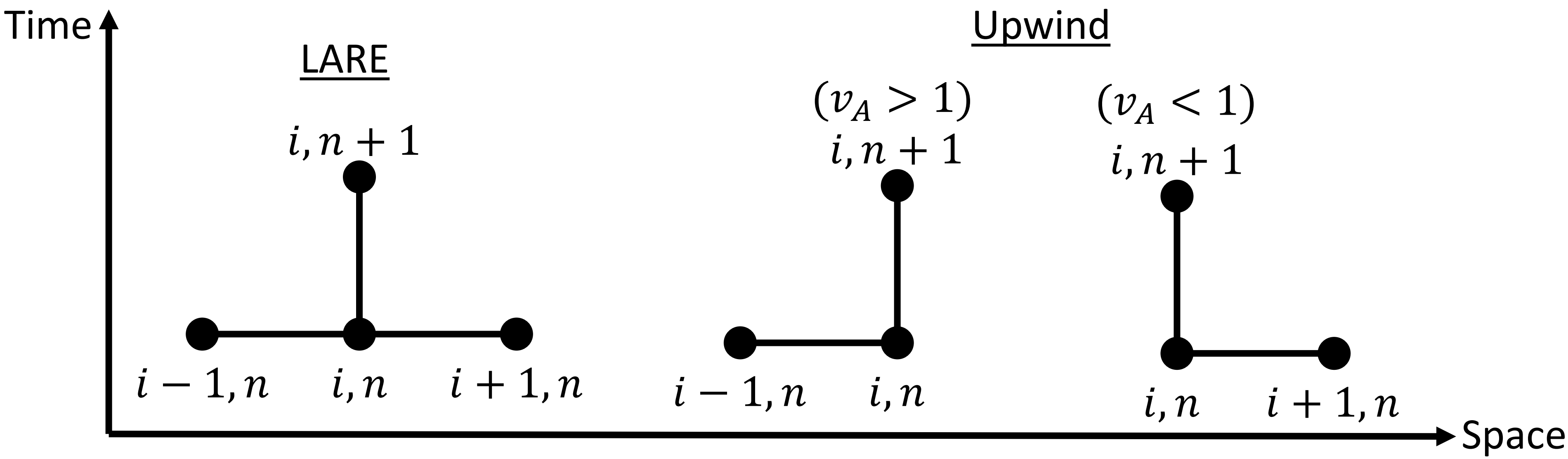
$$u = \sum_{k=0}^{m} (-1)^k R^k H(\theta_k) \sin(\omega\theta_k),$$

$$\theta_k = t - (-1)^k \frac{s}{v_A} - \frac{2k+1}{v_A} l,$$

$$m = \left\lfloor \frac{t v_A}{L} \right\rfloor.$$

Numerical solution

- Difficult to implement in LARE.
- We need an arbitrary amplitude to reflect.
- Decided it would be easier to make a linear upwind code.
- Stencil diagram:



Numerical solution

- Upwind code solves for the Elsässer variables,
$$z^{\pm} = u \pm b.$$
- Elsässer variables satisfy the advection equation,
$$\frac{\partial z^{\pm}}{\partial t} \mp v_A \frac{\partial z^{\pm}}{\partial s} = 0.$$
- Hence, z^{+} travels in the negative direction and z^{-} travels in the positive direction.

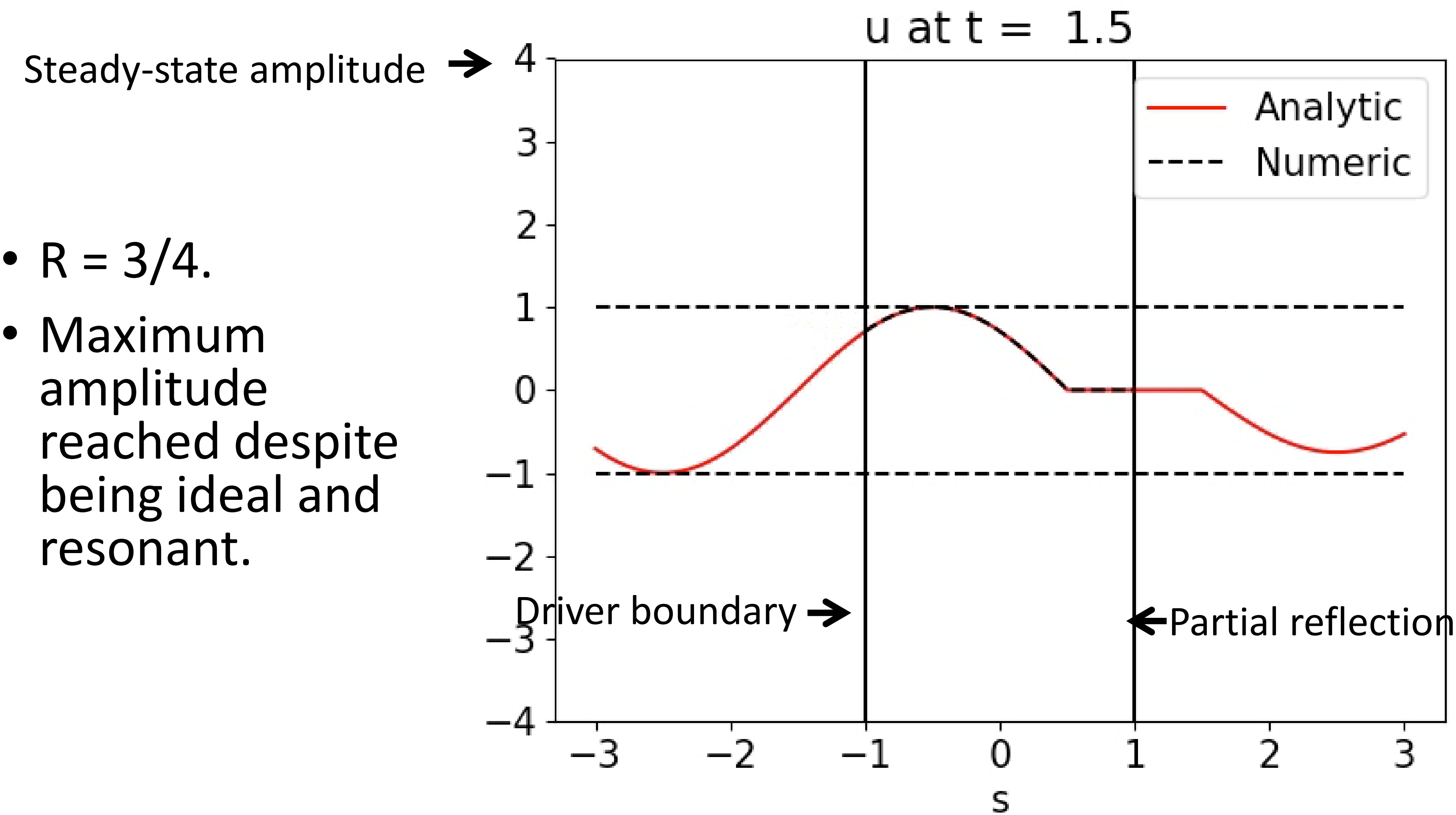
Numerical solution

- To generate a driver, z^- is imposed on the left boundary.
- A factor $\mathrm{z}^+ = Rz^-$ is generated at the right boundary and $\mathrm{z}^-= Rz^+$at the left boundary.
- This simulates partial reflection.

Numeric vs Analytic

Resonant driver

- R = 3/4.
- Maximum amplitude reached despite being ideal and resonant.



Why does the system reach a steady-state?

- Intuitive reason:
  - A wave losing energy through leakage is in some sense equivalent to a wave losing energy via diffusion.
  - Hence, a steady-state is reached for the same reason it does if the wave is damped.
- Mathematical reason:
  - The solution can be rewritten as a geometric series, which converges to a finite value for $R < 1$.

Writing the solution as a geometric series

$$\boxed{\begin{gathered} u = \sum_{k=0}^{m}(-1)^k R^k H(\theta_k)\sin(\omega\theta_k), \\ \theta_k = t - (-1)^k \frac{s}{v_A} - \frac{2k+1}{v_A} l, \\ m = \left\lfloor \frac{t v_A}{L} \right\rfloor, \end{gathered}}$$

let $\mathrm{s} = 0$ and replace $\sin()$ with $\exp(\mathrm{i})$,

$$u = e^{i\omega t - l/v_A} \sum_{k=0}^{m-1}(-1)^k \left(R \cdot e^{-\frac{2i\omega l}{v_A}}\right)^k + (-1)^m R^m H(\theta_m) e^{i\omega\theta_m},$$

$= 0, \mathrm{t} \to \infty$

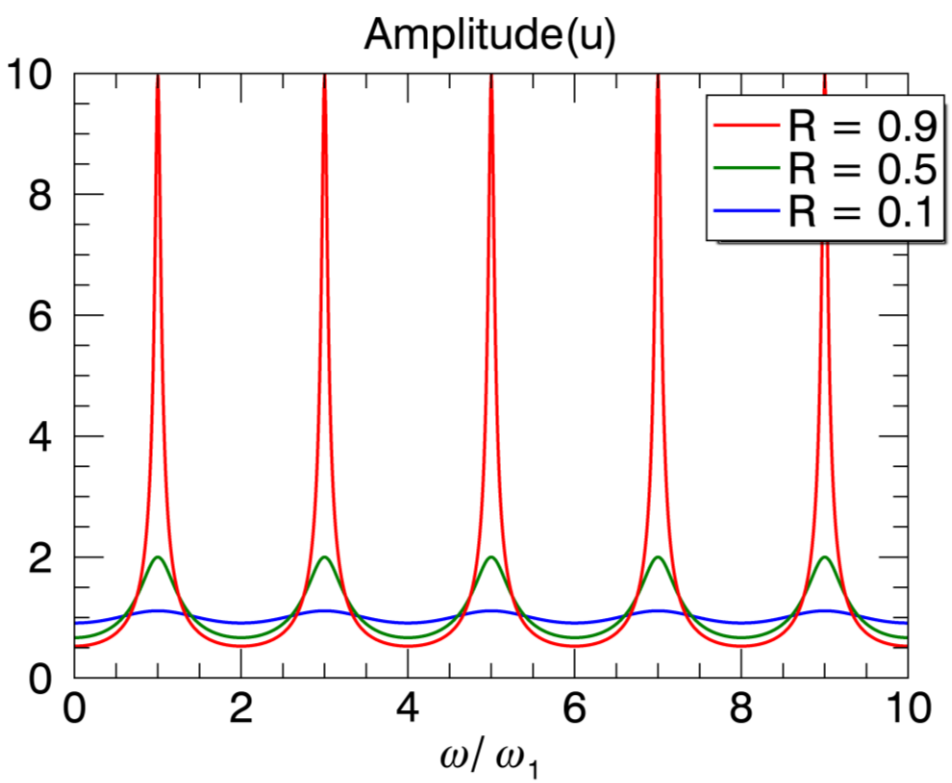
let $\mathrm{t} \to \infty$,

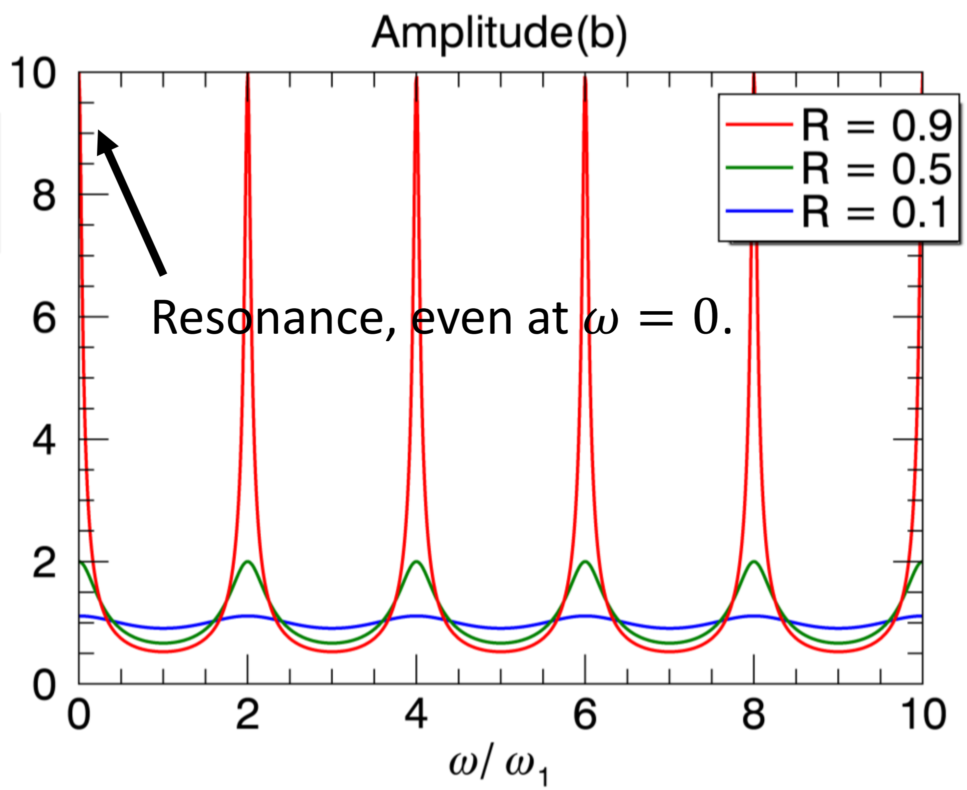
$$\boxed{u(0,t) = \frac{e^{i\omega t}}{1 + Re^{-2i\omega l/v_A}}.}$$

Geometric series (converges for $R < 1$).

Steady-state amplitude at the loop apex

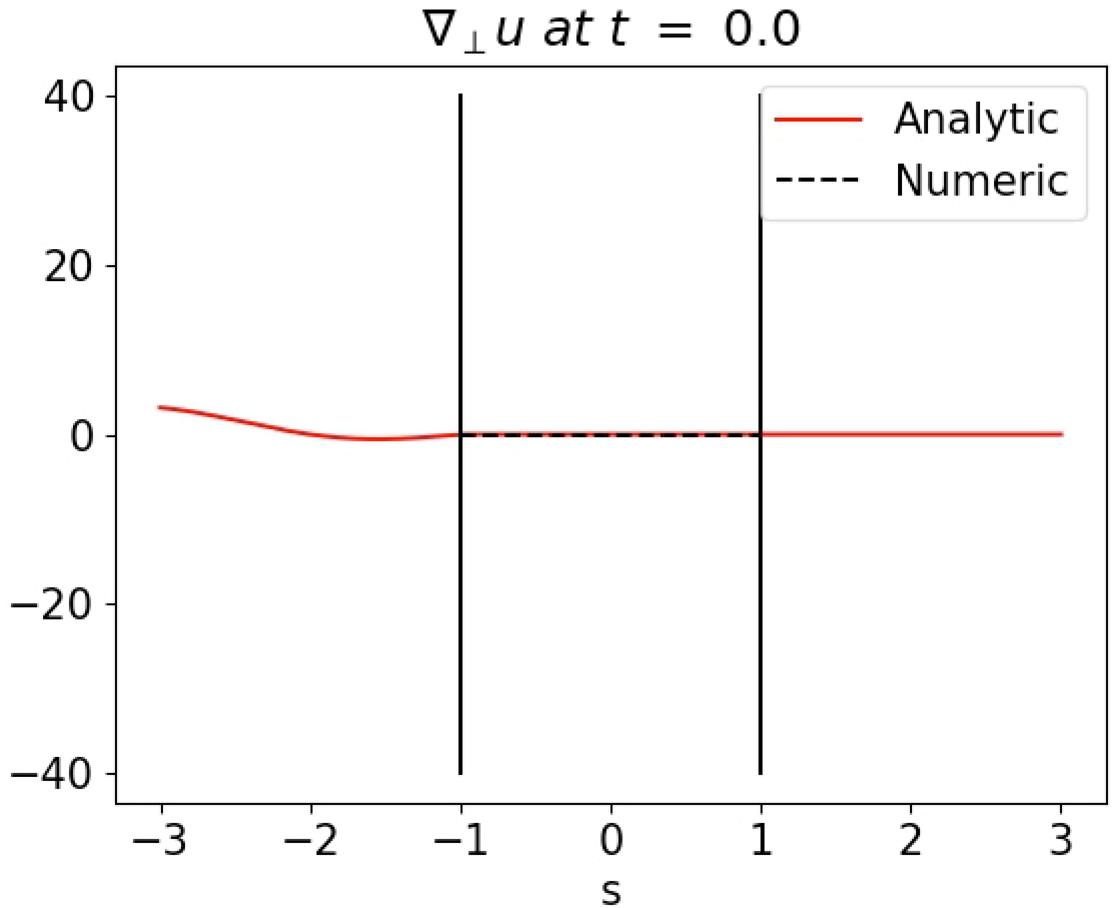
- $\text{Amplitude}(u) = \dfrac{1}{\sqrt{1+2R\cos\left(4\frac{\omega l}{v_A}\right)+R^2}},$
- $\text{Amplitude}(b) = \dfrac{1}{\sqrt{1-2R\cos\left(4\frac{\omega l}{v_A}\right)+R^2}},$
- Convergence timescale: $R^m = \exp\left(\left\lfloor\frac{t\,v_A}{L}\right\rfloor \ln R\right) \Rightarrow \tau \sim \frac{L}{v_A\,|\ln R|}$.





Does $\nabla_{\perp} u$ reach a steady-state?

- Yes (for $R < 1$).
- Same is true for $\nabla_{||} u$.
- Amplitude of discontinuity initially grows linearly then decays exponentially.



Amplitude($\nabla_{\perp}u$) vs Amplitude($\nabla_{||}u$)

- Key point: ∇u (and ∇b) tends towards a steady-state.
- Since the plasma is ideal, phase mixing is solely responsible for the build-up in $\nabla_{\perp}$.
- If we can relate $\nabla_{\perp}u$ to $\nabla_{||}u$ e.g.
$$\nabla_{\perp}u = C\nabla_{||}u$$
- Then we know phase mixing has enhanced the gradients by a factor C.

Amplitude($\nabla_\perp u$) vs Amplitude($\nabla_{||} u$)

At steady-state

$$\mathrm{Amp}\,(\nabla_\perp u) = \frac{l}{v_A} \nabla_\perp v_A \left| \frac{1 - R\exp\left(-i\,\frac{\omega L}{v_A}\right)}{1 + R\exp\left(-i\,\frac{\omega L}{v_A}\right)} \right|^2 \mathrm{Amp}\left(\nabla_{||} u\right).$$

$$\frac{1-R}{1+R} \leq \left| \frac{1 - R\exp\left(-i\,\frac{\omega L}{v_A}\right)}{1 + R\exp\left(-i\,\frac{\omega L}{v_A}\right)} \right| \leq \frac{1+R}{1-R}$$

Driven at even harmonic.

Driven at odd harmonic.

$\mathrm{Amplitude}(\nabla_\perp u)$ vs $\mathrm{Amplitude}(\nabla_{||} u)$

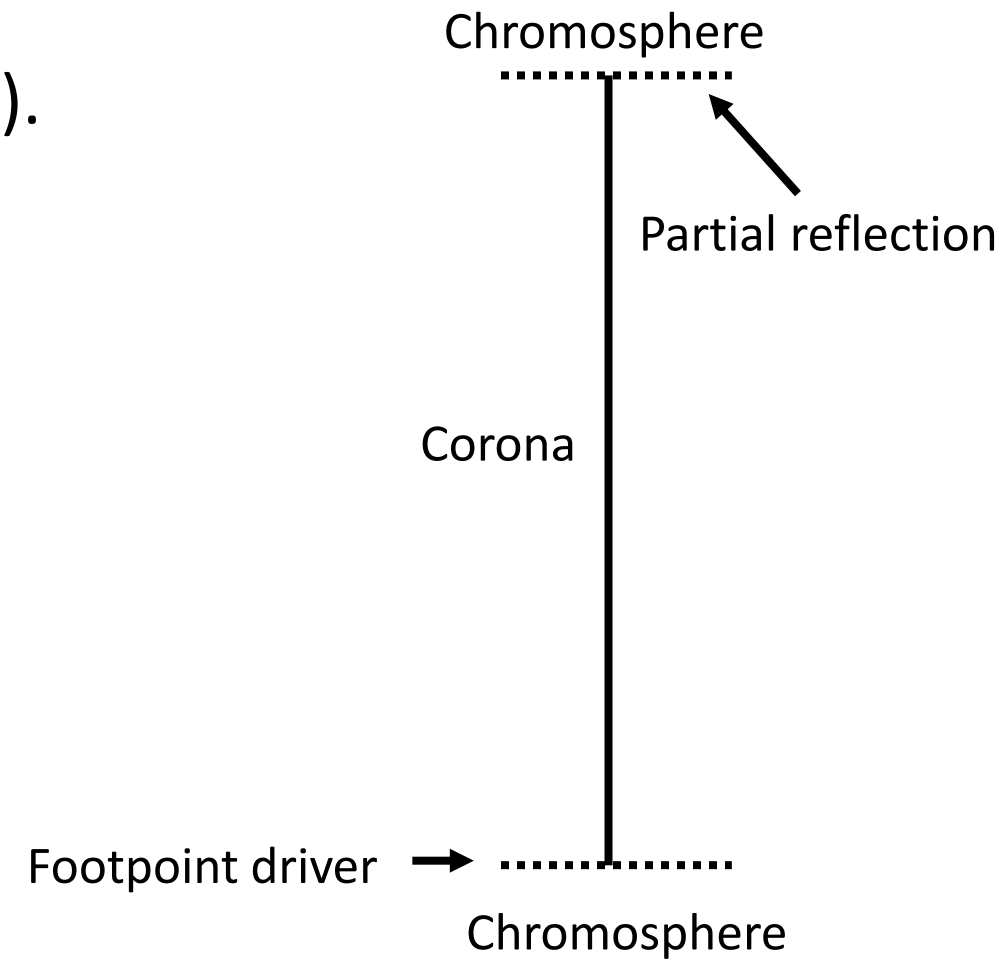
At steady-state

$$10^{-1}\mathrm{Amp}\left(\nabla_{||}u\right) \leq \mathrm{Amp}\ (\nabla_\perp u) \leq 10^4\ \ \mathrm{Amp}\left(\nabla_{||}u\right).$$

- If:
  - $l = 100\ \mathrm{Mm}$,
  - $v_A = 400\ \mathrm{km\ s^{-1}}$,
  - $\nabla_\perp v_A = 100\ \mathrm{km\ s^{-1}\ Mm^{-1}}$,
  - $R = 9/10$.
- Using a broadband driver could help reduce this range.

Experiment 3/3

- Parital reflection ($R < 1$).
- Broadband driver.



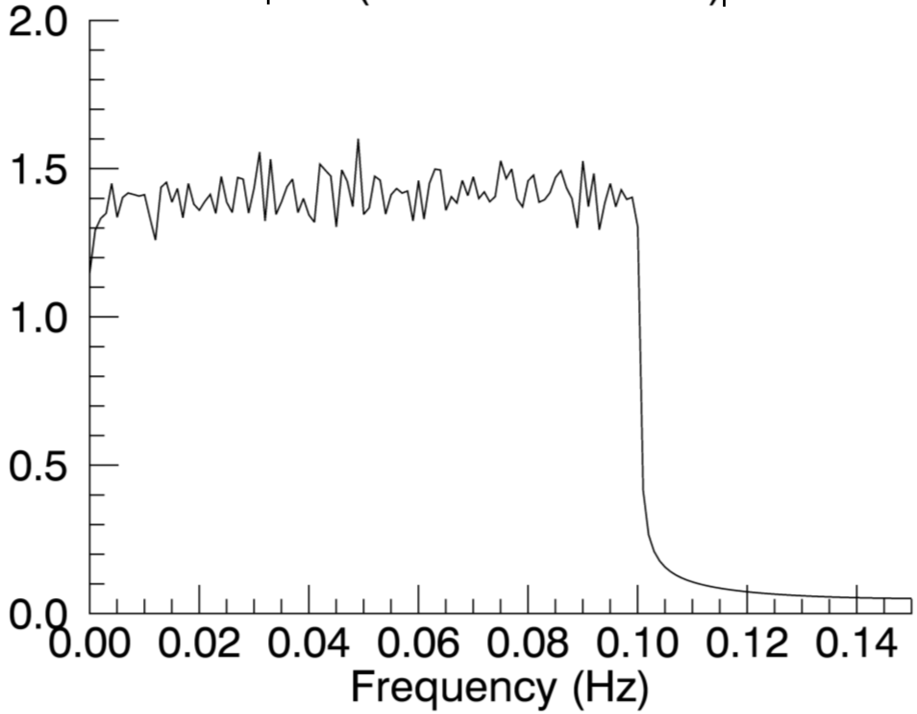
What is a broadband driver?

- A broadband driver is a driver which excites a range of frequencies.

$$u = \sum_{n=0}^{N} \sin(\omega_n t + \phi_n)$$

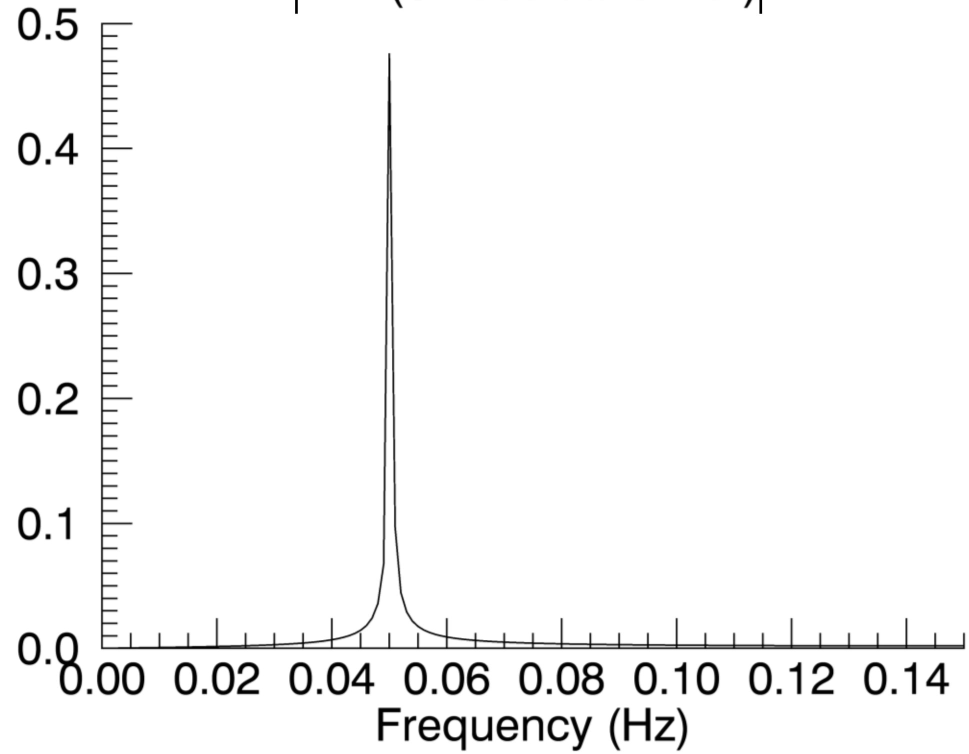
Random phase.

|FFT(Broadband driver)|



$$u = \sin(\omega t)$$

|FFT(Sinusoidal driver)|



Analytic solution

- We have a solution for a sinusoidal driver, at steady state:

$$u(0,t) = \mathrm{Amp}(\omega, R)\sin(\omega t)\,.$$

- Our driver now has the form:

$$u_{driv}(t) = \sum_{n=1}^{N} f_n^{-\alpha} \sin(\omega_n t + \phi_n)\,.$$

Gives $\sqrt{\text{power spectrum.}}$

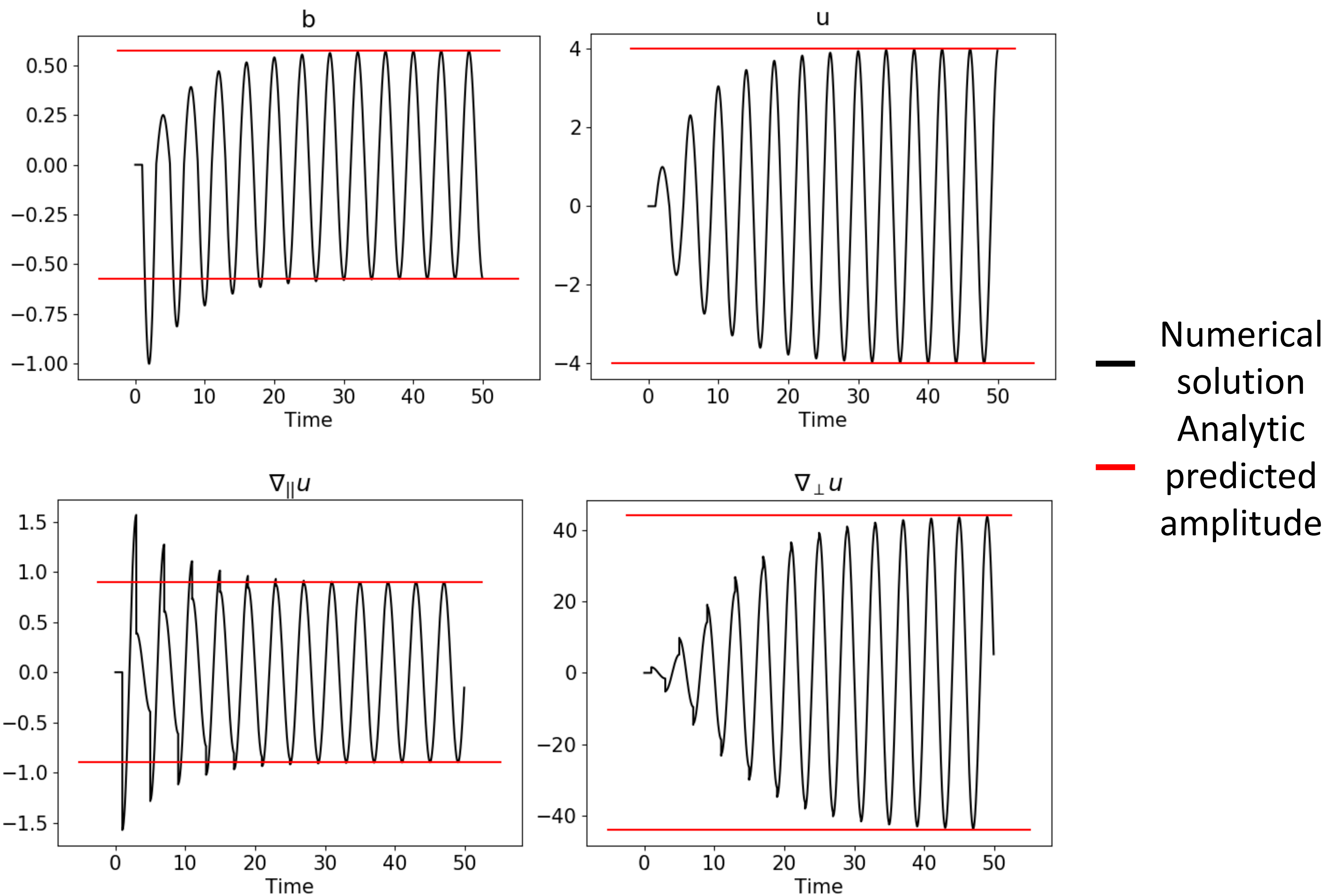
- So the solution is:

$$u(0,t) = \sum_{n=1}^{N} f_n^{-\alpha} \mathrm{Amp}(\omega_n, R)\sin(\omega_n t + \phi_n)\,.$$

Why no numerical solution?

- Not yet had time.
- We have checked the analytic formula for sinusoidal drivers.
- Linear problem $\Longrightarrow$ solution is a superposition of the sinusoidal driver experiments.

Formula check



The driver

- Takes the form,

Gives $\sqrt{\text{power spectrum}}$.

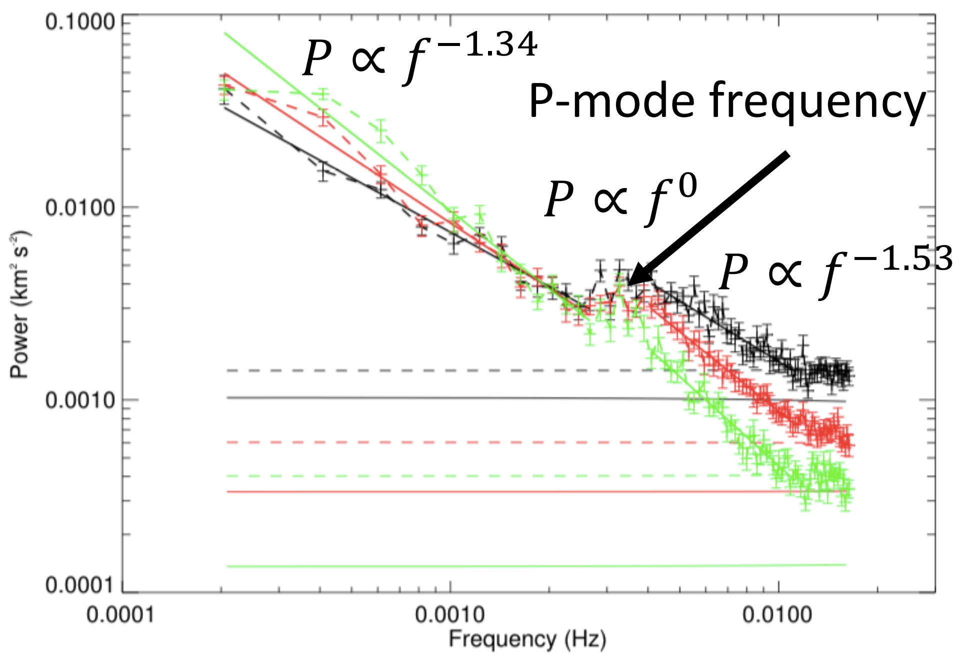
$$u_{driv}(t) = \sum_{n=1}^{N} \mathrm{f}_n^{-\alpha} \sin(\omega_n t + \phi_n)\,.$$

Can take any value.

- $N = 1000.$
- $f_n = \frac{\omega_n}{2\pi} = n10^{-4}$ Hz
- ϕ_n is a random phase with a uniform probability distribution over the interval $[0,2\pi]$.

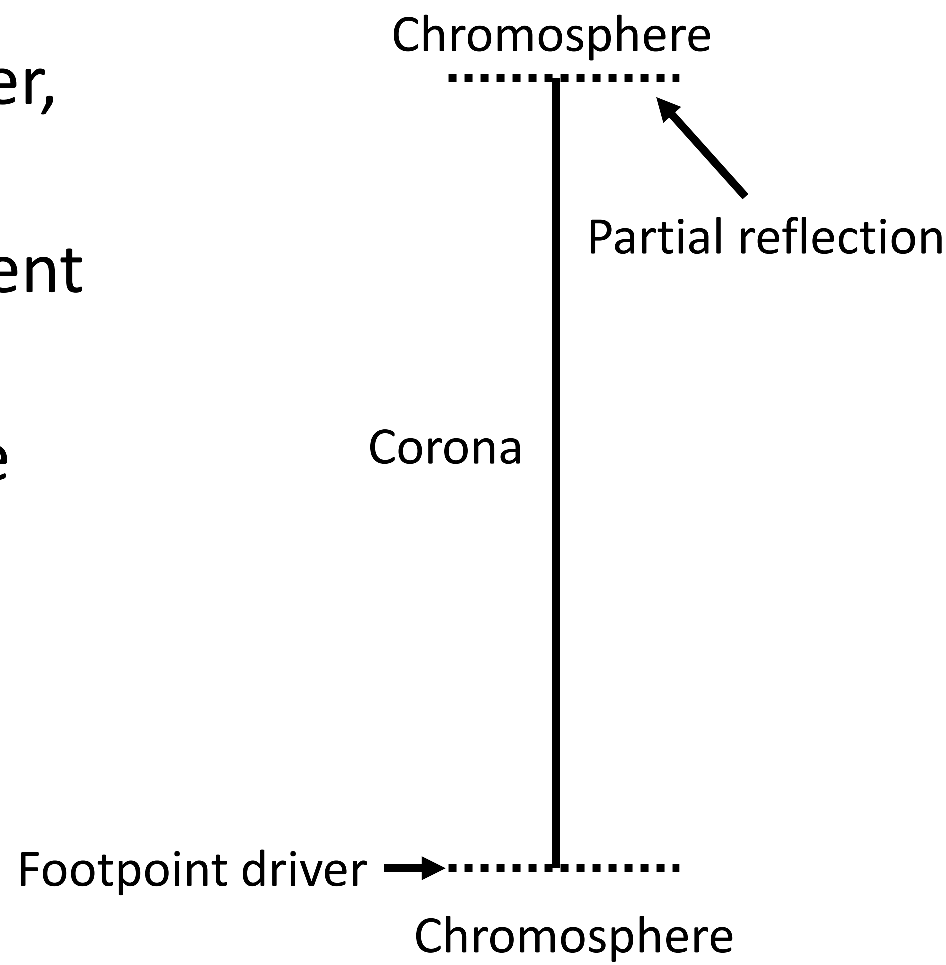
The driver

- Each term is multiplied by a factor $f_n^{-\alpha}$.
- $\alpha = [0.1, 0.5, 0.9]$ depending on the experiment.
- Since $u \propto \sqrt{P}$ we can see that $0 < \alpha < 1$.
- Results are mainly insensitive to our choice of α.



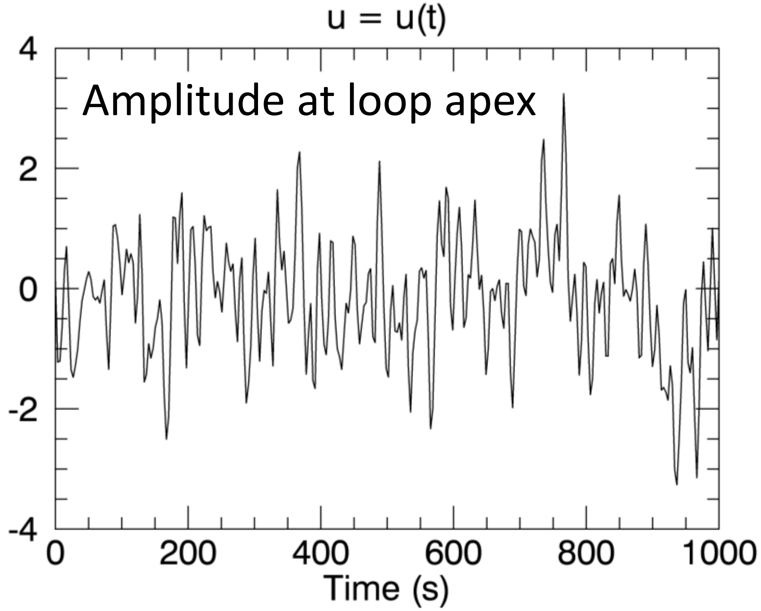
Experiment 3 - summary

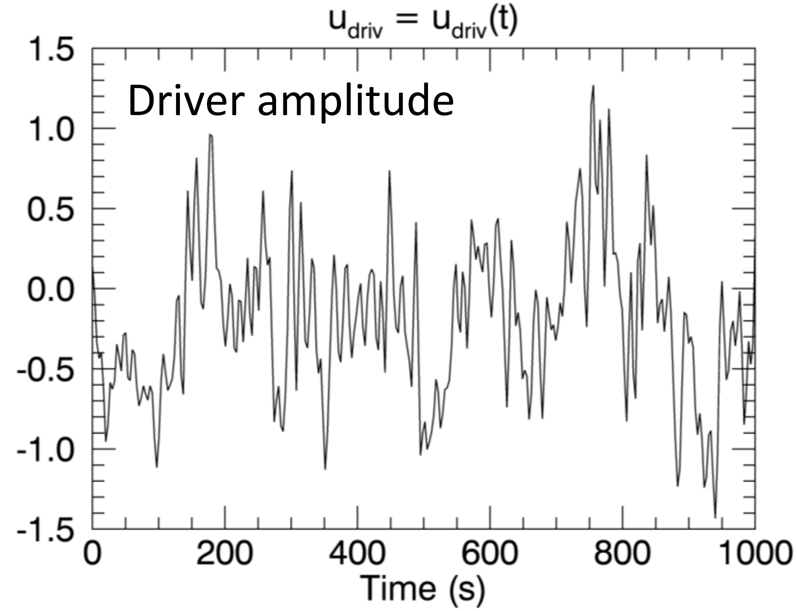
- Drive with a broadband driver, with a random phase.
- Repeat 1000 times and present the average.
- Use the analytic steady-state formulas.
- $L = 200\text{Mm}.$
- $v_A = 1\ \text{Mm s}^{-1}$

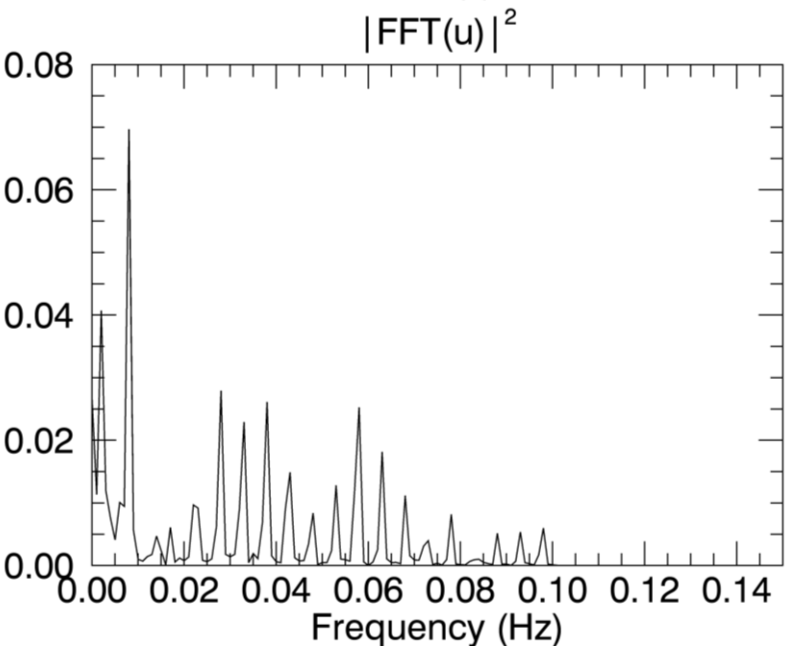


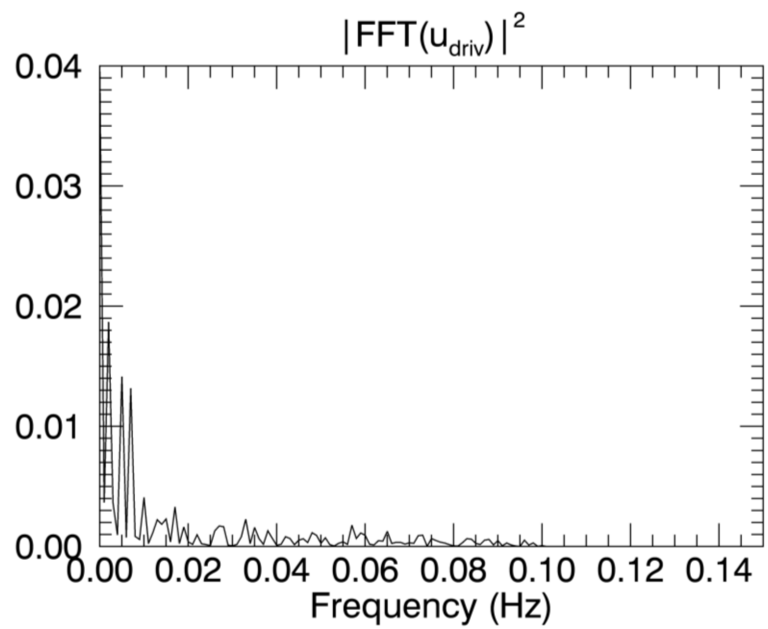
$R = 0.9, \alpha = 0.5$

Random experiment



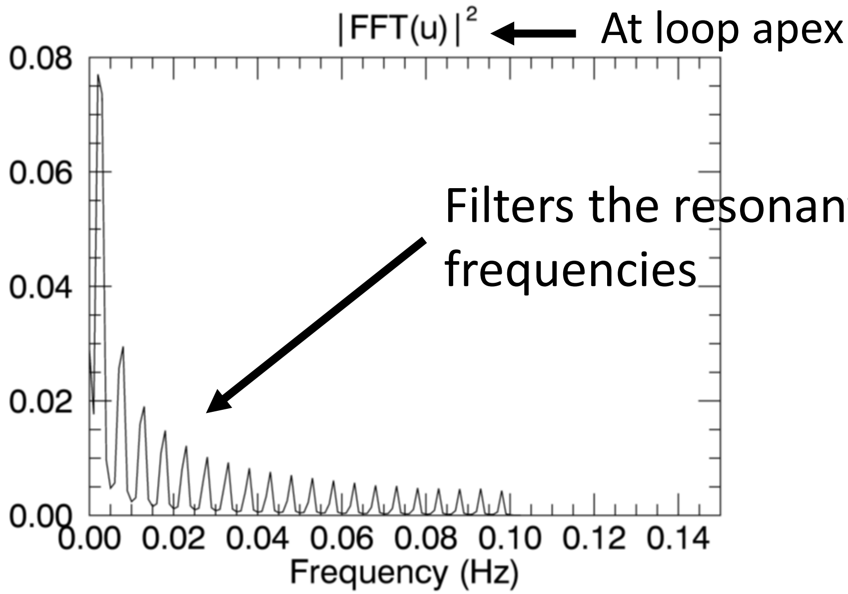


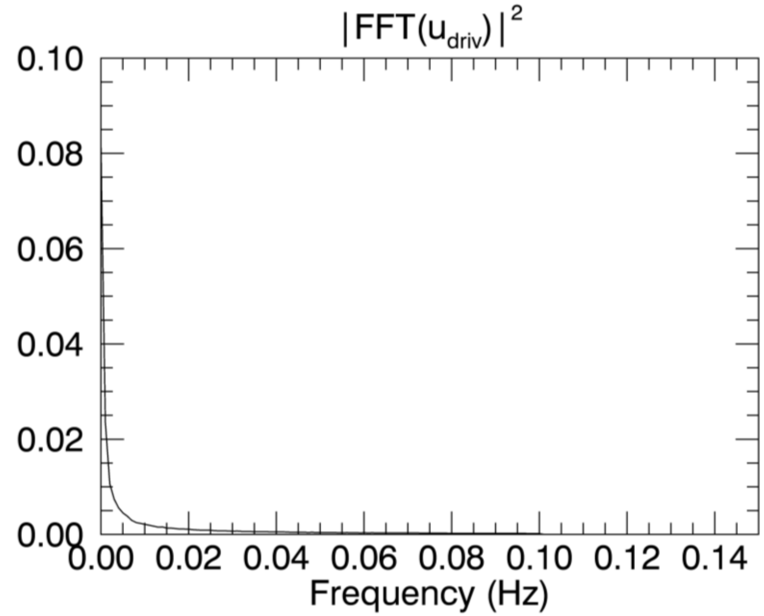


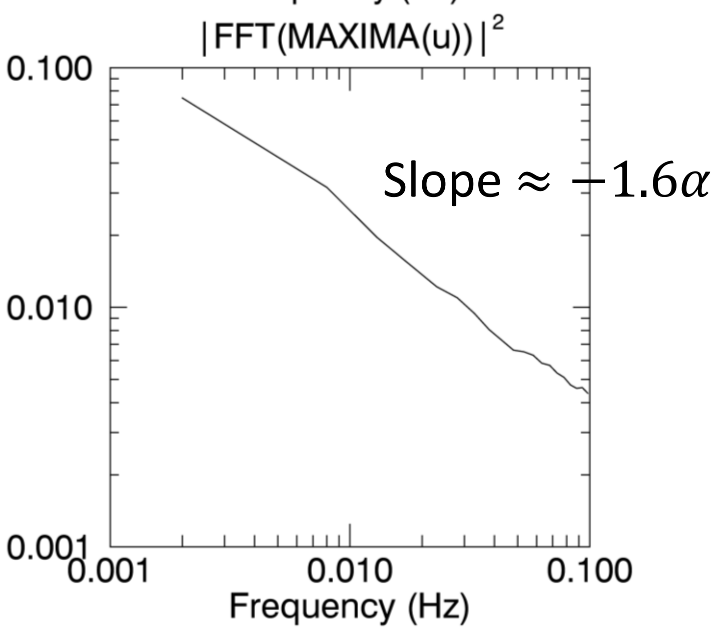


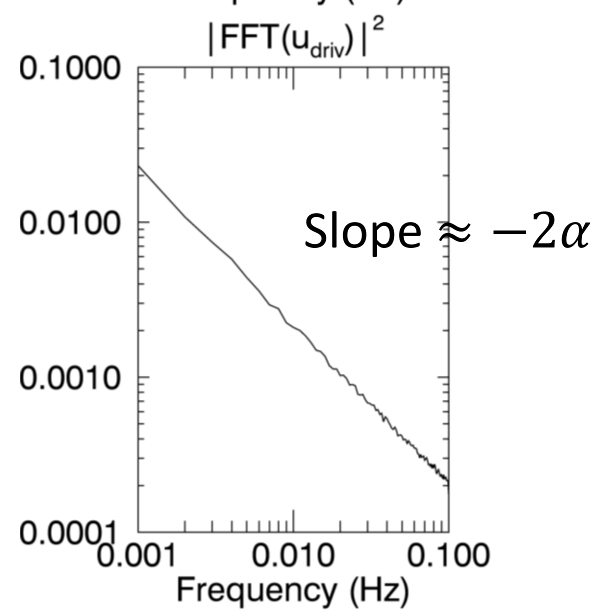
$R = 0.9, \alpha = 0.5$

Average over 1000 experiments

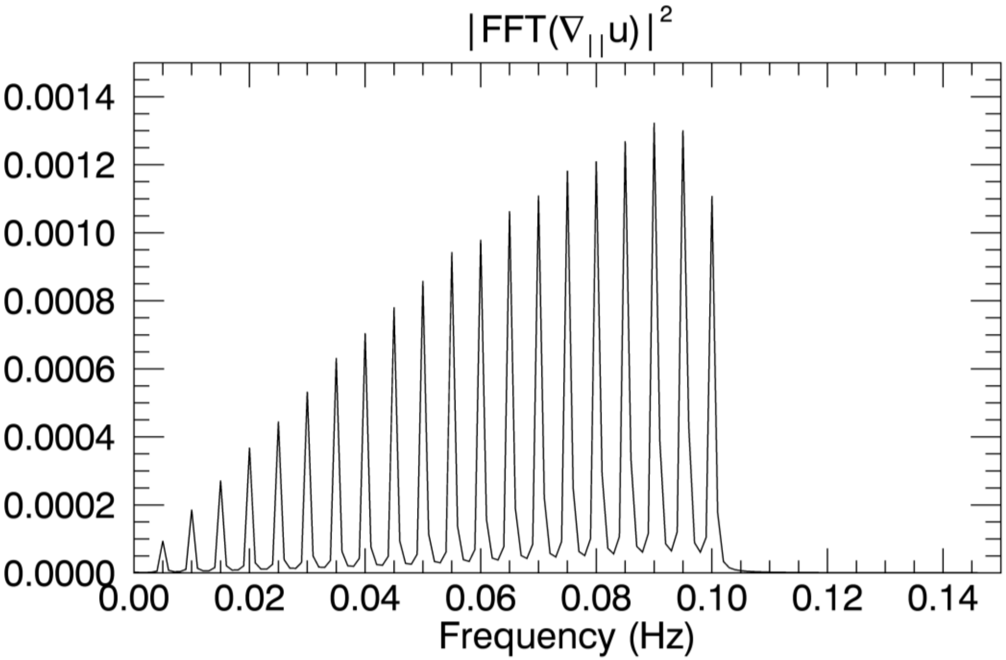


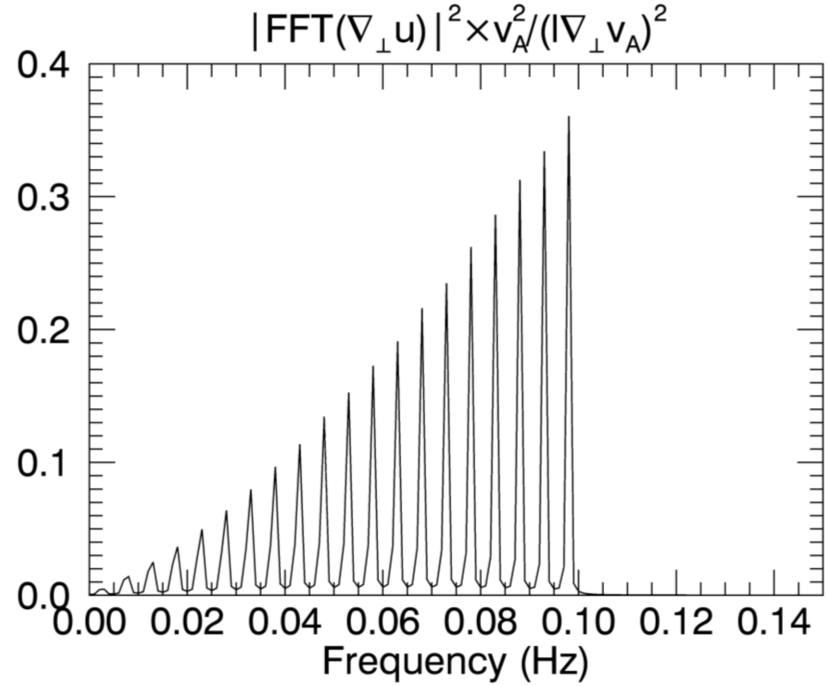






$R = 0.9, \alpha = 0.5$
Average over 1000 experiments





- Higher frequencies have the highest power.
- Sinusoidal driver:

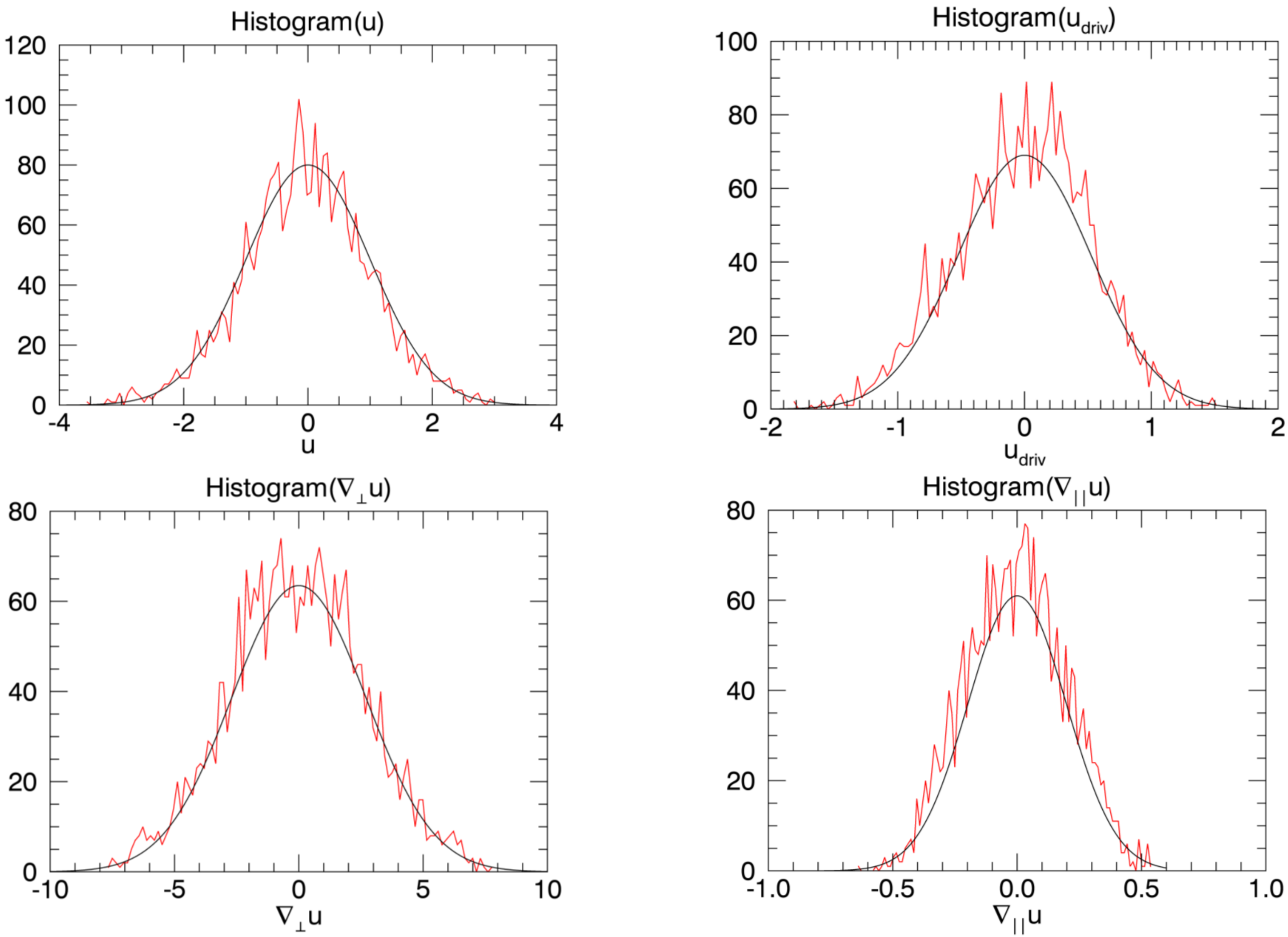
$$\nabla_{||}u \propto \omega$$

- Broadband driver:

$$u_{driv,n} \propto \omega_n^{-\alpha}$$
$$\Longrightarrow \nabla_{||}u \propto \omega_n^{1-\alpha}$$

Values are ≈ normally distributed

Random experiment, $R = 0.9, \alpha = 0.5$



Standard deviation of $\nabla_{||} u$ vs. $\nabla_{\perp} u$

- STD = standard deviation.
- Table of the average $\mathrm{STD}(\nabla_{\perp} u)/\mathrm{STD}(\nabla_{||} u)$:

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	Analytic
$R = 0.5$	2.11	2.16	2.18	3
$R = 0.9$	14.0	13.7	13.8	19
$R = 0.99$	191	192	195	199
$R = 0.999$	1930	1950	1980	1999

- Proposed analytic formula (not proven):

$$\mathrm{STD}(\nabla_{\perp} u) \approx \frac{l}{v_A} \nabla_{\perp} v_A \frac{1+R}{1-R} \mathrm{STD}(\nabla_{||} u)$$

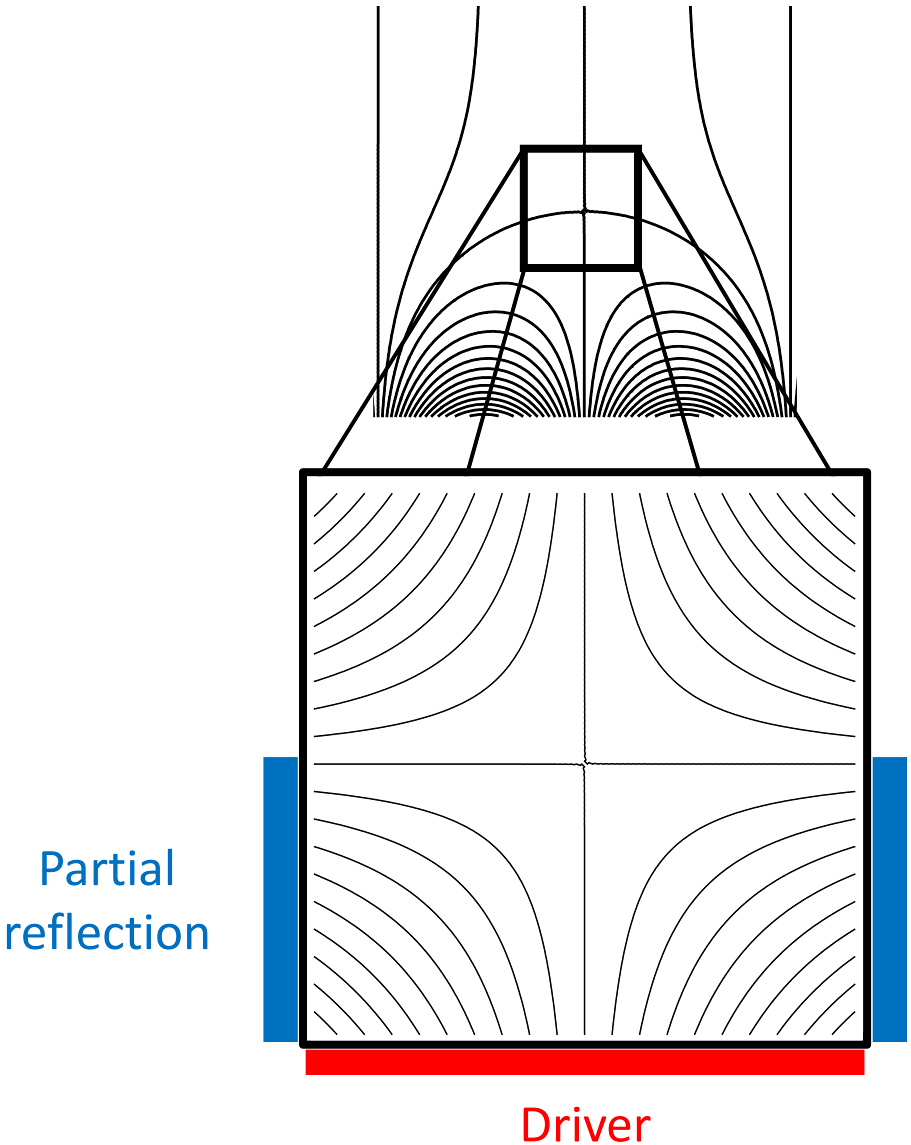
Simplifications

- We have a formula relating $\nabla_{\perp} u$ and $\nabla_{||} u$.
- Assumed ideal MHD, therefore, our relation can be seen as an upper bound for $\nabla_{\perp} u$.
- However, these simplifications were also made:
  - Linear waves.
  - Consider only strict Alfvén waves.
  - $\nabla_{||} v_A = 0^*$.
- Future work could look at at the effects of modifying these simplifications.

Application of $\nabla_\perp u$ and $\nabla_{||} u$ relation

X-point field

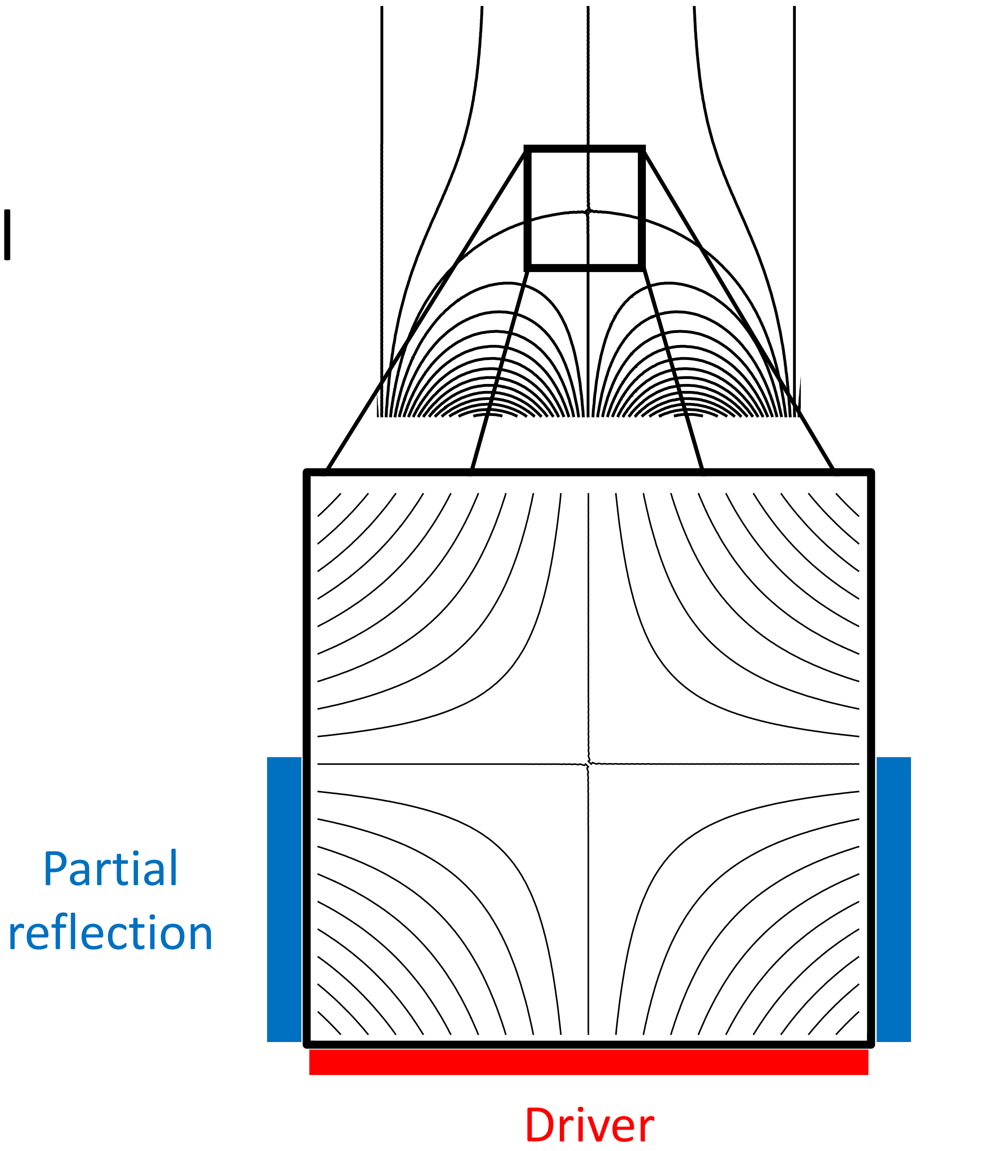
- Goal from previous work: Look at phase mixing due to variations in field line length.
- Formula could help deduce a parameter space where the phase mixing is negligible.



Application of $\nabla_{\perp} u$ and $\nabla_{||} u$ relation

X-point field

- Wave equation for ideal Alfvén waves in a potential field:
$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\mu \rho_0} \left(\vec{B}_0 \cdot \nabla\right)^2 u.$$
- Convert to a field aligned coordinate system to get the 1D wave equation:
$$\frac{\partial^2 u}{\partial t^2} = v_{A0}^2 \frac{\partial^2 u}{\partial s^2}$$

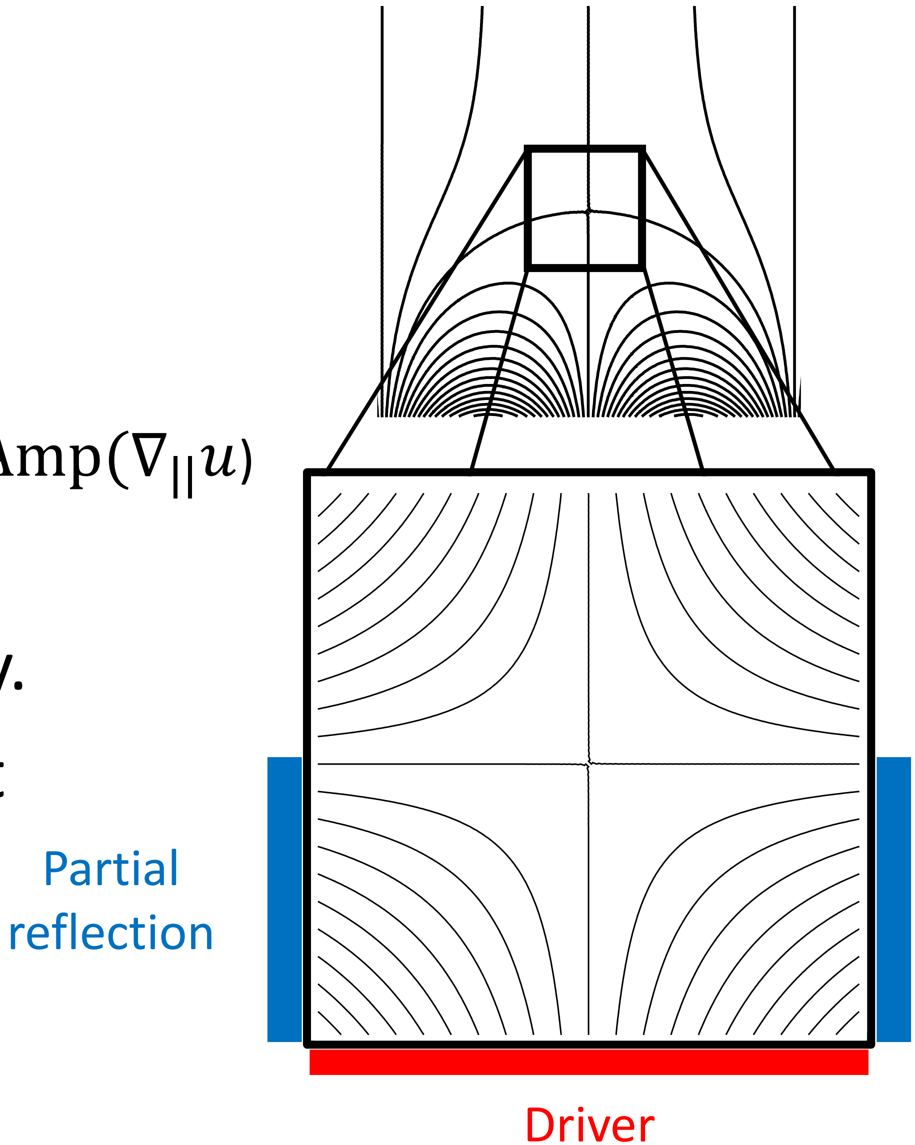


Application of $\nabla_\perp u$ and $\nabla_{||} u$ relation
X-point field

- Calculated for a sinusoidal driver that:

$$\mathrm{Amp}(\nabla_\perp u) = \left| \frac{1 - R\exp\left(-\frac{2i\omega l}{v_A}\right)}{1 + R\exp\left(-\frac{2i\omega l}{v_A}\right)} \right|^2 \mathrm{Amp}(\nabla_{||} u)$$

- Not confirmed numerically.
- Equation is not dependent on normalising constants.



Summary

- We have derived a formula which links $\nabla_{\perp}$ and $\nabla_{||}$.
- In ideal MHD, phase mixing is only responsible for enhancing $\nabla_{\perp}$.
- Therefore, it is hoped that our relation can be seen as an upper bound for the enhancement in heating due to phase mixing.

Future work

- Can we prove a formula for the relationship between $\mathrm{STD}\left(\nabla_{||} u\right)$ and $\mathrm{STD}(\nabla_{\perp} u)$?
- Investigate the effects of switching to:
  - A nonlinear system.
  - 3D.
  - Other modes of oscillation.

Thank you for listening

Questions?