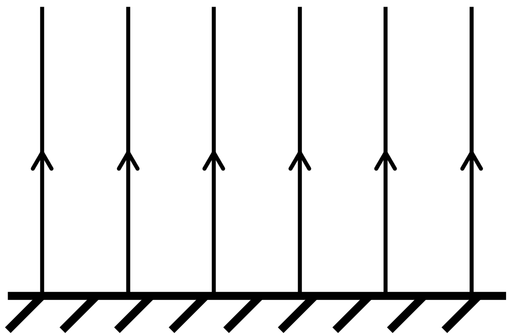
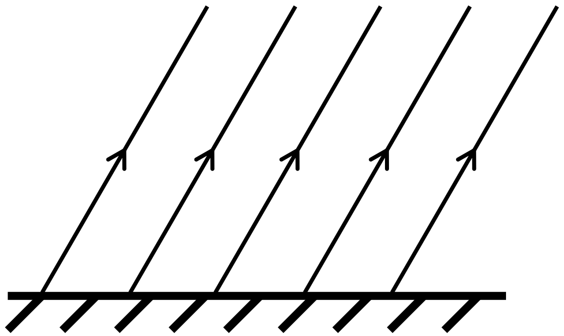
Resonant absorption in an oblique magnetic field

Alex Prokopyszyn, Alan Hood, Andrew Wright

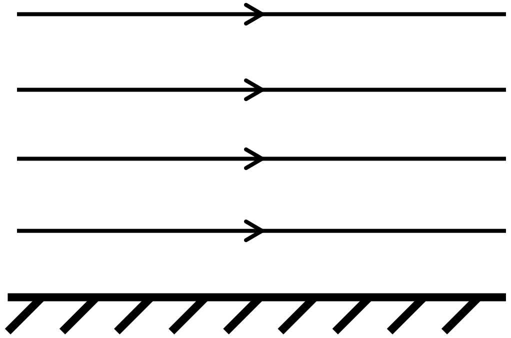
Perpendicular field



Oblique field



Parallel field



Aim

- Study resonant absorption in an oblique field

Methods

- Analytic and numerical

Results

- Steep boundary layers form
- → limited effect on the rate of absorption

What is resonant absorption?

- Process where:
  - Magnetoacoustic waves → standing Alfvén waves
  - $\omega_f = \omega_n$

Magnetoacoustic wave frequency

Alfvén frequency

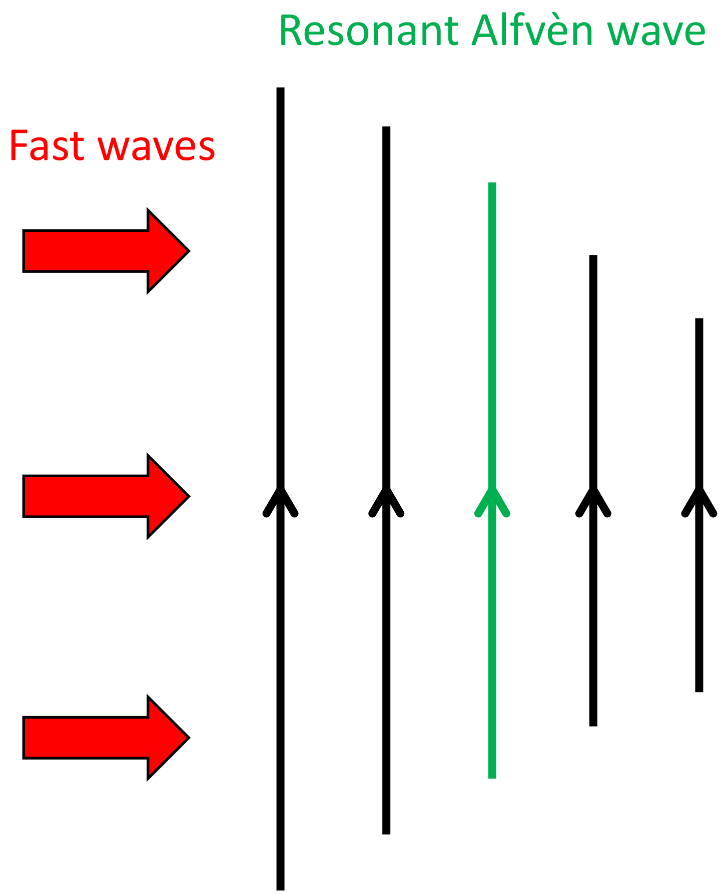
- Analogous to Barton's pendulum →

Courtesy youtube.com/Anthony_Francis-Jones

What is resonant absorption?

Barton’s pendulum

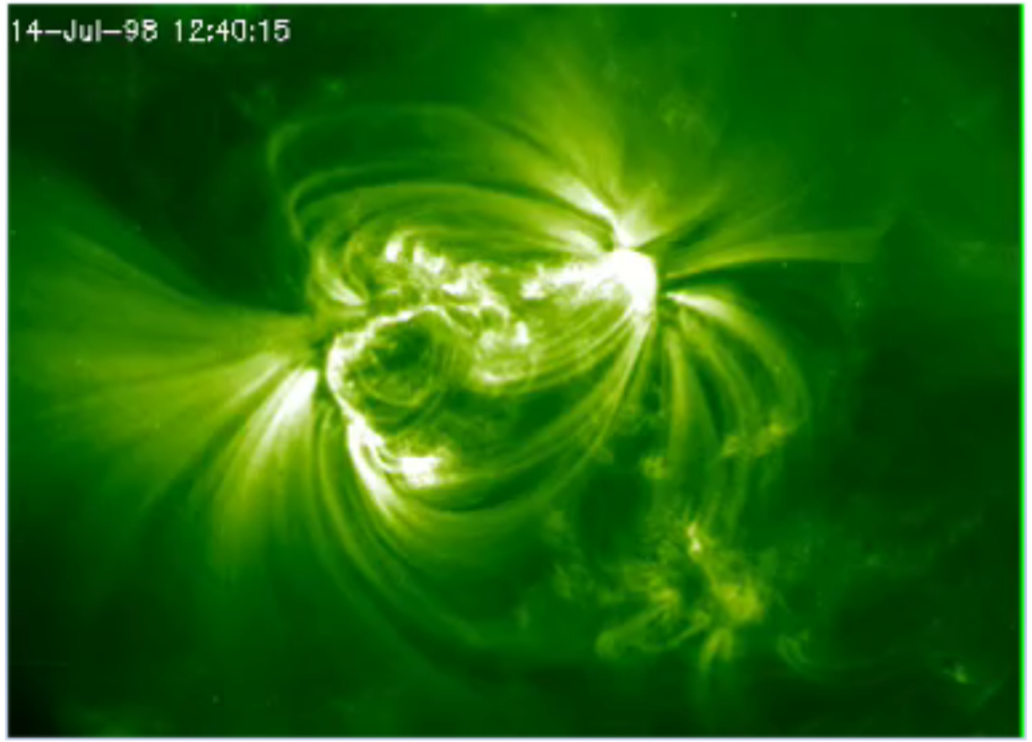
~



Resonant absorption

Why study resonant absorption?

- Coronal seismology
- Explains why waves damp so fast (Ruderman and Roberts, 2002)
- → short length-scales ⟹ heating



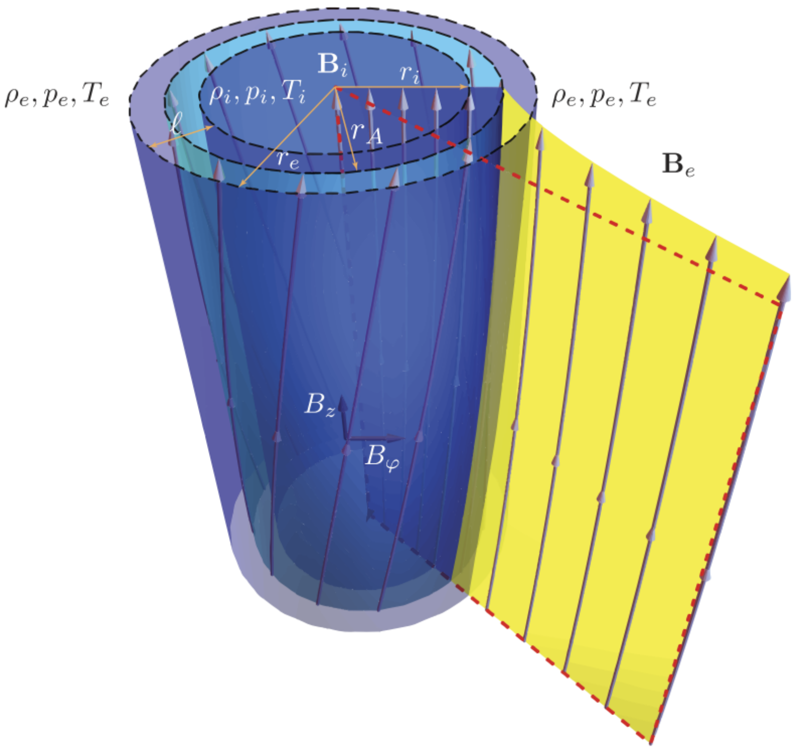
Video captured using TRACE at 171 Å

Why study an oblique field?

- Solid wall + oblique field → boundary layers

- Implications for coronal seismology are not obvious

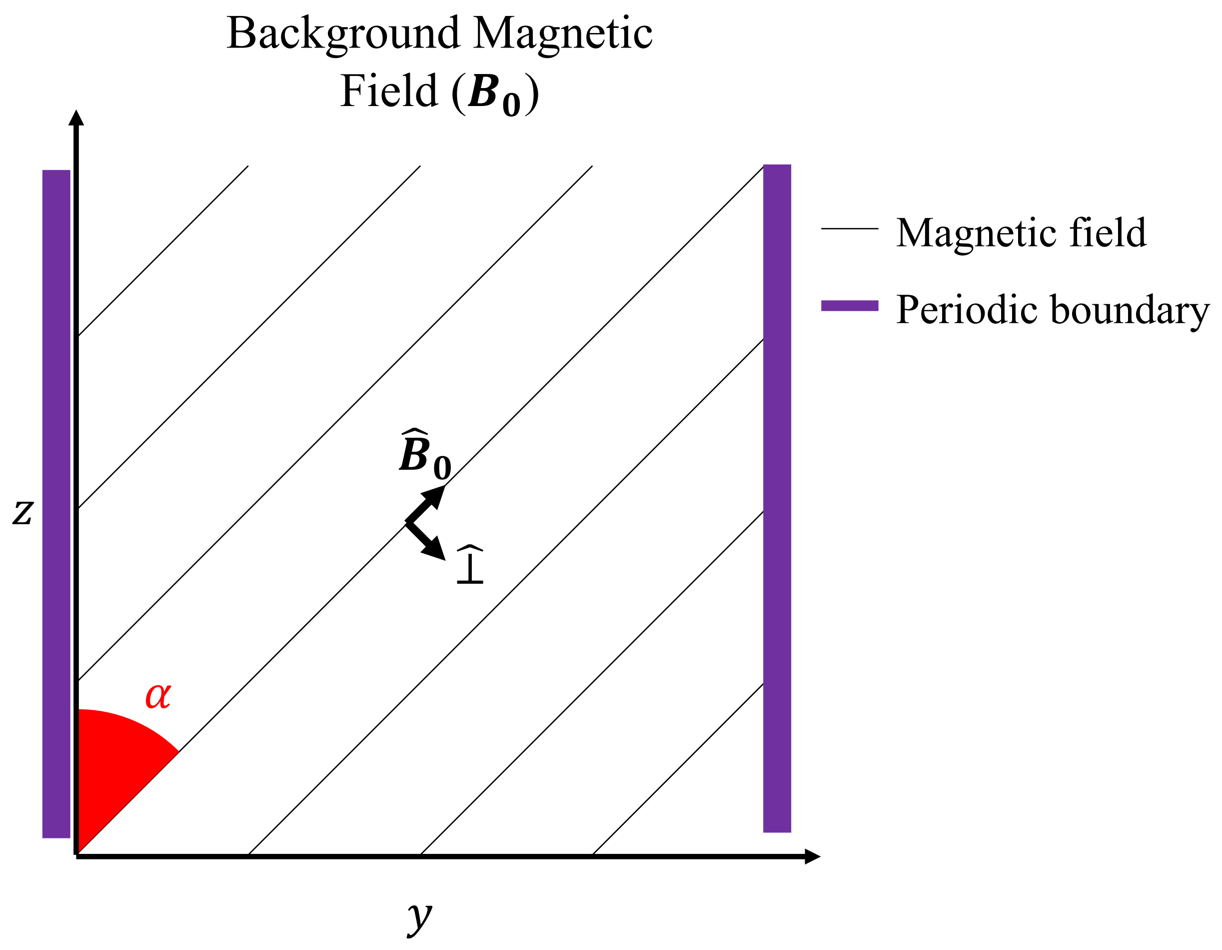
- Solid wall + oblique field has not been studied in detail in a RA context



Courtesy Giagkiozis et al. (2016)

Our model

- Background quantities:
  - $\boldsymbol{B}_0 = B_0 \widehat{\boldsymbol{B}}_0$
  - $v_A(x) = v_{A0}(1 - x/L_z)$
- Perturbations:
  - $\boldsymbol{u}(x, y, z, t) = u_x \widehat{\boldsymbol{x}} + u_\perp \widehat{\perp}$
  - $\boldsymbol{b}(x, y, z, t) = b_x \widehat{\boldsymbol{x}} + b_\perp \widehat{\perp} + b_{||} \widehat{\boldsymbol{B}}_0$
- Where:
  - $\widehat{\boldsymbol{B}}_0 = \sin\alpha\ \widehat{\boldsymbol{y}} + \cos\alpha\ \widehat{\boldsymbol{z}}$
  - $\widehat{\perp} = \cos\alpha\ \widehat{\boldsymbol{y}} - \sin\alpha\ \widehat{\boldsymbol{z}}$



MHD Equations

- Ideal, $\beta = 0$, linear

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{1}{\rho} \boldsymbol{j} \times \boldsymbol{B}_0$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}_0)$$

$$\nabla_{\perp} = \cos\alpha \ \frac{\partial}{\partial y} - \sin\alpha \frac{\partial}{\partial z}$$

$$\nabla_{||} = \sin\alpha \ \frac{\partial}{\partial y} + \cos\alpha \frac{\partial}{\partial z}$$

$$\Longrightarrow$$

$$\frac{\partial u_x}{\partial t} = \frac{B_0}{\mu\rho}\left[\nabla_{||} b_x - \frac{\partial b_{||}}{\partial x}\right]$$

$$\frac{\partial u_{\perp}}{\partial t} = \frac{B_0}{\mu\rho}\left[\nabla_{||} b_{\perp} - \nabla_{\perp} b_{||}\right]$$

$$\frac{\partial b_x}{\partial t} = B_0 \nabla_{||} u_x$$

$$\frac{\partial b_{\perp}}{\partial t} = B_0 \nabla_{||} u_{\perp}$$

$$\frac{\partial b_{||}}{\partial t} = -B_0\left[\frac{\partial u_x}{\partial x} + \nabla_{\perp} u_{\perp}\right]$$

Normal mode

- We seek normal mode solutions:

$$u_x, u_\perp, b_x, b_\perp, b_\parallel \propto \exp[\mathrm{i}(k_y y + \omega t)]$$

- Where:

$$|k_y L_z| \leq O(1) \in \mathbb{R}$$

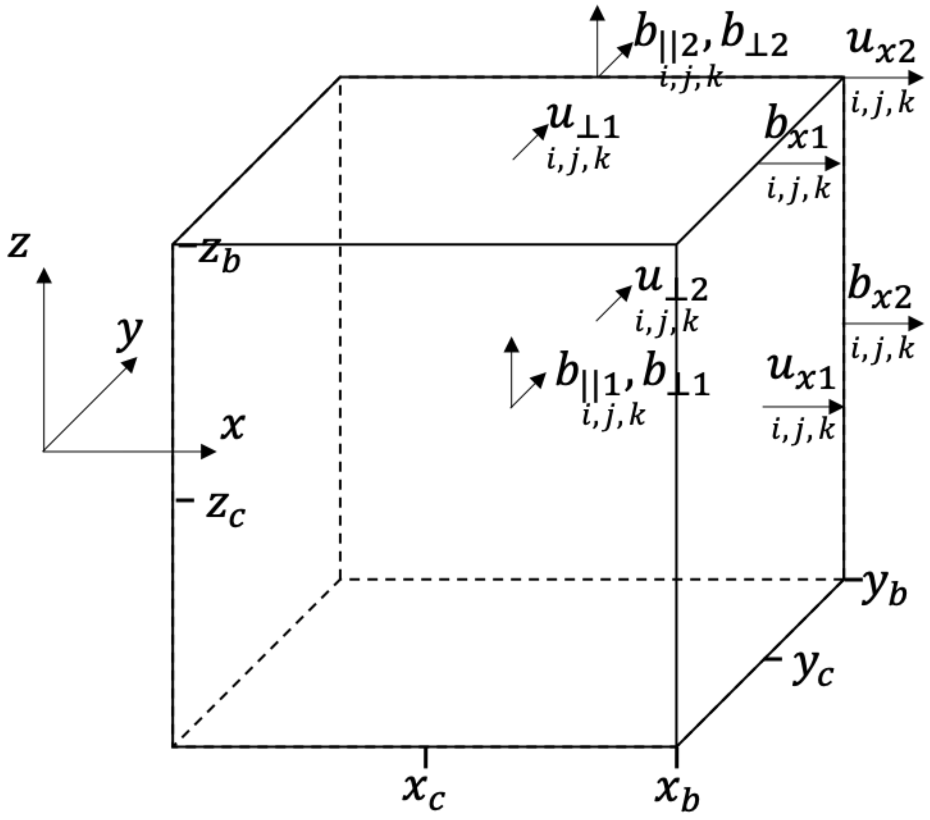
$$\omega = \omega_r + i\omega_i$$

$$\omega_r = \frac{\pi v_{A0}}{L_z} \cos\alpha$$

$$\frac{\omega_i}{\omega_r} = \epsilon \ll 1$$

Numerical schemes

- Solve ideal, $\beta = 0$, linear MHD equations
- Leapfrog algorithm:
  - Based on Rickard and Wright (1994)
  - Finite-difference
  - Staggered grid
  - Second order accurate
- Runge-Kutta algorithm:
  - Solves ODE which we derive later



Structure

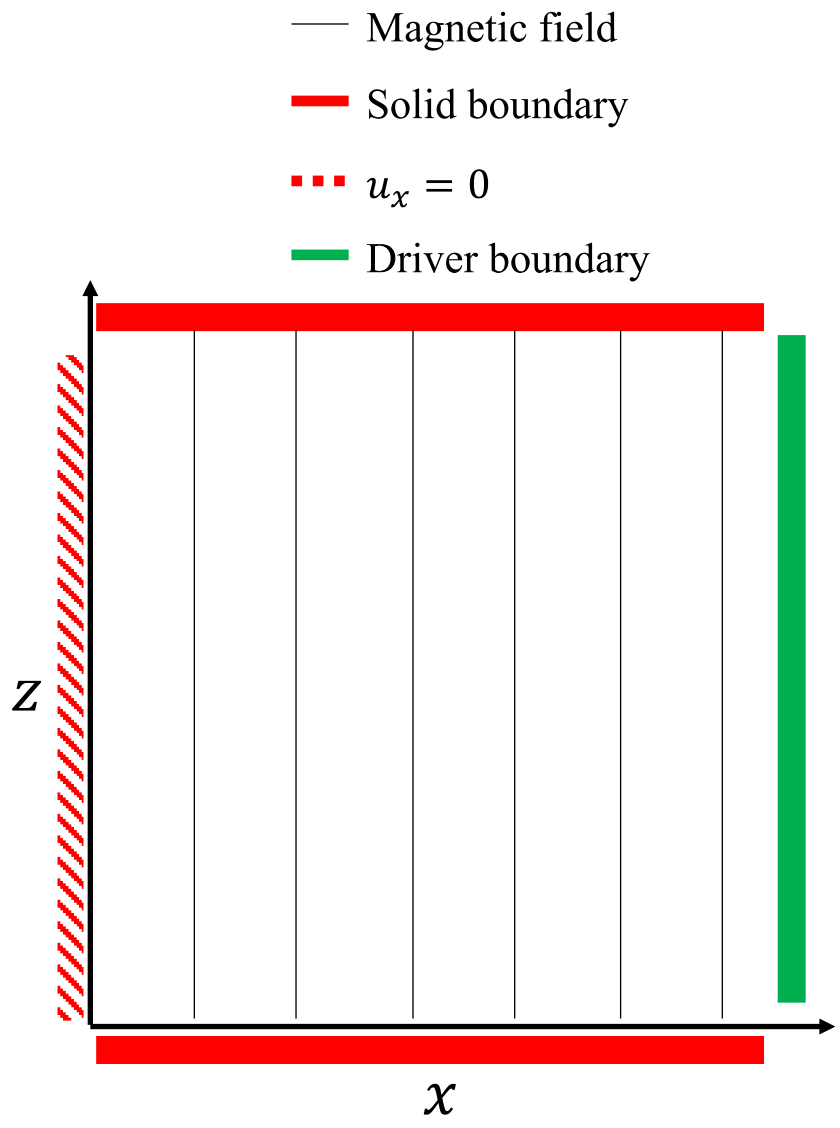
- Introduction
- RA in a perpendicular field
- Uniform domain
- Normal mode solution
- Solution with BCs
- Summary

Boundary conditions

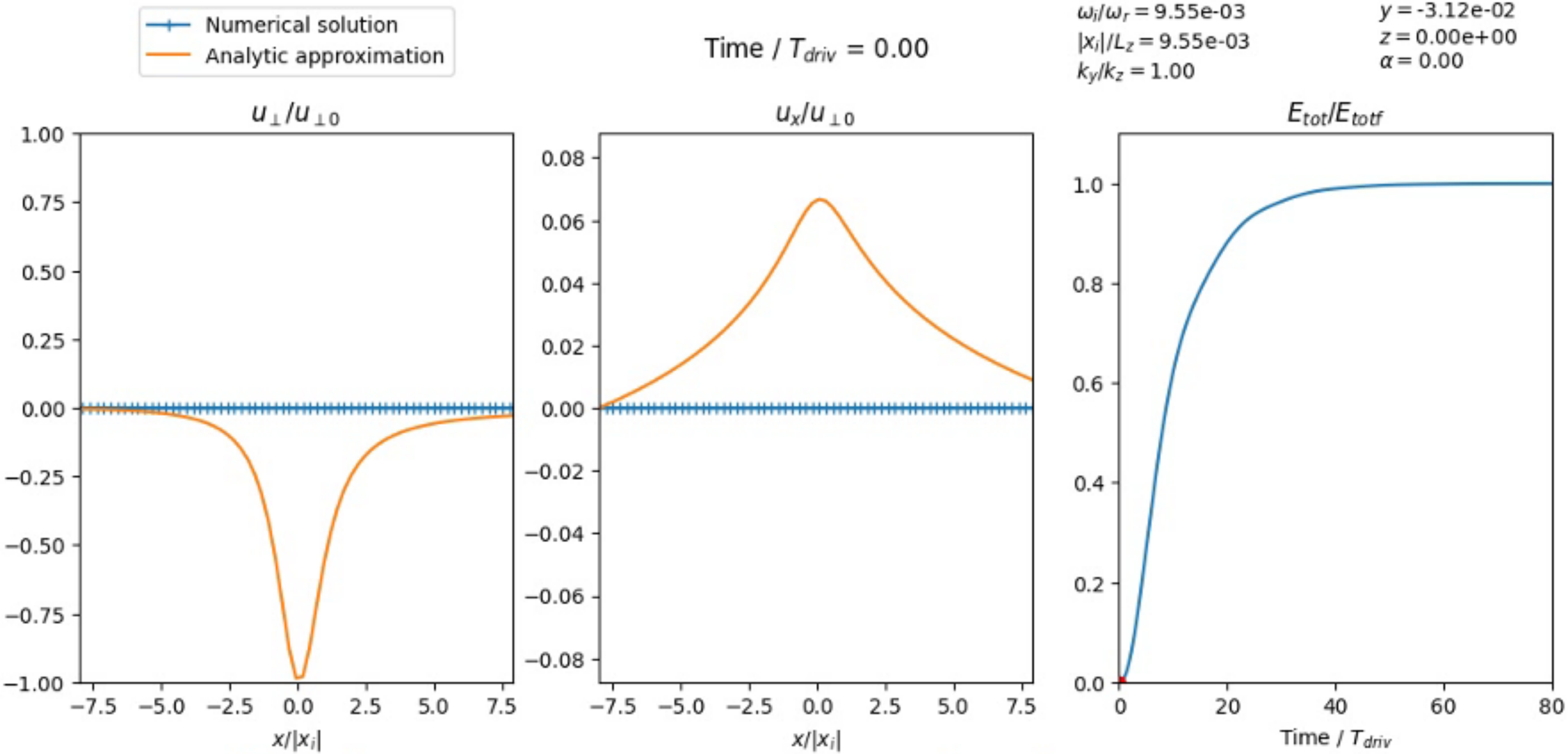
- Driver at $x = x_{max}$, given by:

$$b_{\parallel} = \sin(k_y y - \omega_r t)\cos(k_z z)\exp(\omega_i t)$$

- Periodic in y
- $\boldsymbol{u} = 0$ at $z = z_{min}$ and z_{max}
- $u_x = 0$ at $x = x_{min}$
- Solve using leapfrog code

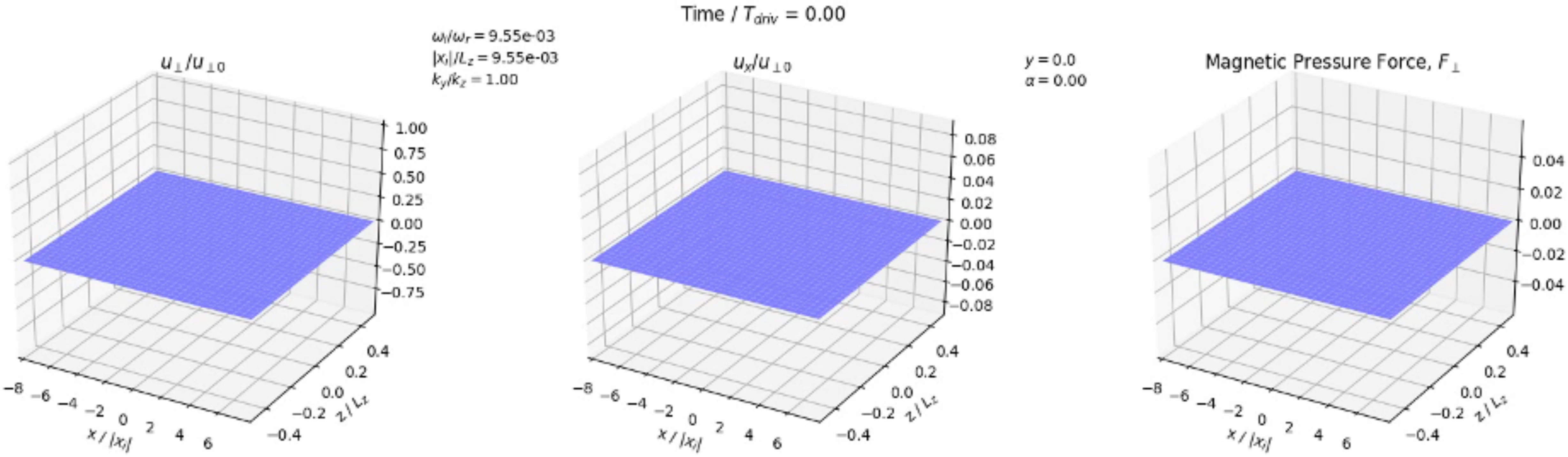


$$u_{\perp 0}(t) = \max(u_\perp)\exp(\omega_i t)$$
$$E_{totf}(t) = \max(E_{tot})\exp(\omega_i t)$$



$u_\perp/u_{\perp 0} \approx \mathrm{Re}(x_i e^{i(k_y y - \omega t + \phi)}/(x - ix_i))\cos(k_z z)$, $u_x/u_{\perp 0} \approx \mathrm{Re}(-ik_y x_i e^{i(k_y y - \omega t + \phi)}[\ln(x - ix_i) + C])\cos(k_z z)$

$$u_{\perp 0}(t) = \max(u_{\perp}) \exp(\omega_i t)$$



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Why study waves in a uniform domain?

- $\rho = \rho_0$
- Provides simplest demonstration of the boundary layers forming
- Can calculate exact analytic solution
- Test code

Normal mode solution

- Assume:

$$u_x, u_\perp, b_x, b_\perp, b_{||} \propto \exp[i(k_x x + k_y y + k_z z + \omega t)]$$

- Where:

$$k_x, k_y, \omega \in \mathbb{R} \text{ and } k_z \in \mathbb{C}$$

Dispersion relation

- Convert PDE's into an algebraic equation:

$$\Longrightarrow \underbrace{\left[\omega^2 + v_A^2 \nabla_{\parallel}^2\right]}_{\text{Alfvén waves}} \underbrace{\left[\omega^2 - v_A^2\left(k_x^2 + k_y^2 + k_z^2\right)\right]}_{\text{Fast waves}} = 0$$

- Where: $\nabla_{\parallel} = i(k_y \sin\alpha + k_z \cos\alpha)$
- Assume: $v_A^2\left(k_x^2 + k_y^2\right) > \omega^2$
- Hence:

$$k_{z1} = \frac{\omega}{v_A \cos\alpha} - k_y \tan\alpha$$

$$k_{z2} = -\frac{\omega}{v_A \cos\alpha} - k_y \tan\alpha$$

$$k_{z3} = i\frac{\sqrt{v_A^2\left(k_x^2 + k_y^2\right) - \omega^2}}{v_A}$$

$$k_{z4} = -i\frac{\sqrt{v_A^2\left(k_x^2 + k_y^2\right) - \omega^2}}{v_A}$$

Example: Semi-infinite domain

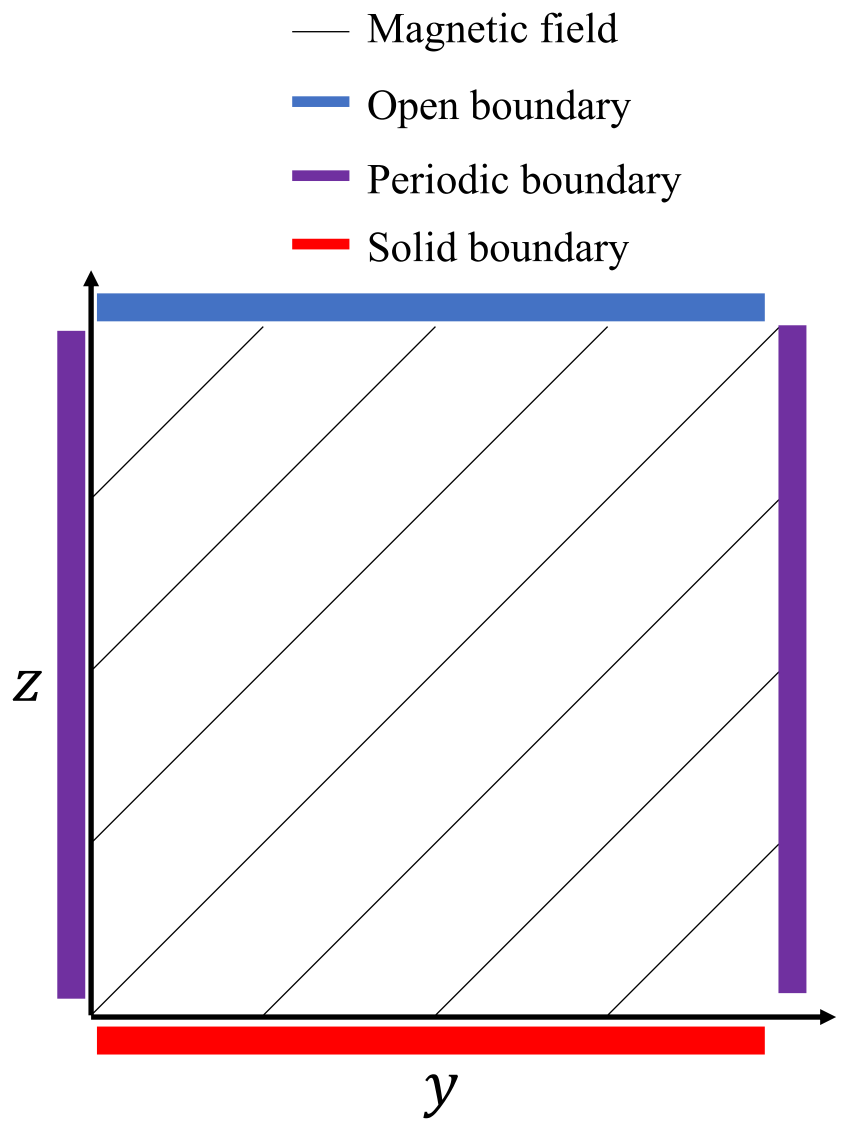
- $z \in [0, \infty)$

- Impose solid BC's at $z = 0$, i.e.

$$\boldsymbol{u}\big|_{z=0} = 0$$

- Open BC's for $z \to \infty$

- Solution remains finite $\forall\, z$



Calculate full solution

- Solution remains finite $\forall\, z \Longrightarrow$ coefficient of k_{z4} term $= 0$
- Boundary conditions $\Longrightarrow u_x = u_\perp = 0$ at $z = 0$
- Specify incident wave amplitude

$\Longrightarrow$ 3 equations 3 unknowns

$\Longrightarrow$ Exact solution

Large k_x approximation

$$u_x(z) = O(k_x^{-1})$$

$$u_\perp(z) = 2i\, I \sin(k_z^* z) \exp\left(-ik_y \tan\alpha\ z\right) + O(k_x^{-1})$$

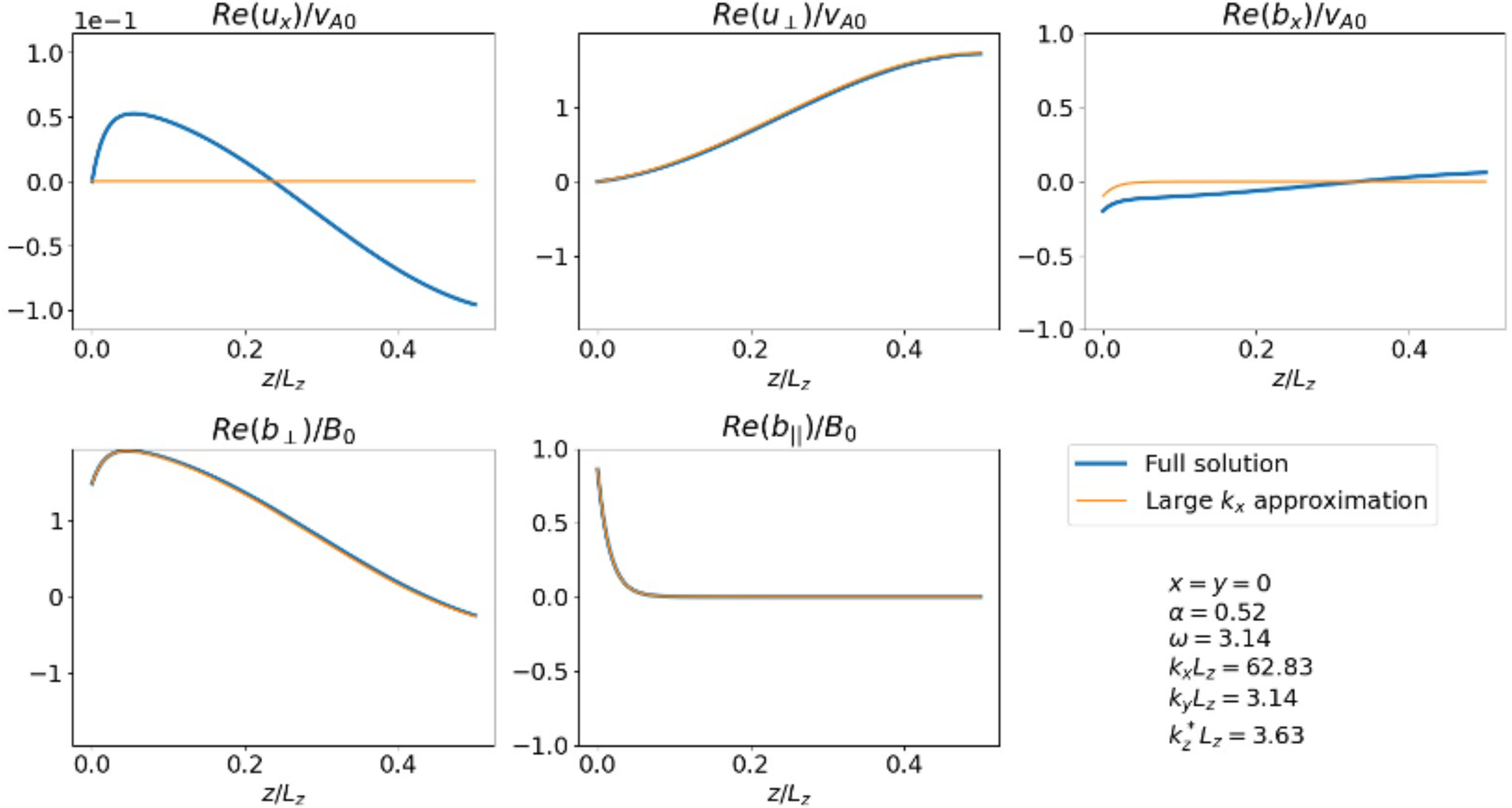
$$b_x(z) = -2iB_0 \frac{I}{v_A} \sin\alpha \exp(-k_x z) + O(k_x^{-1})$$

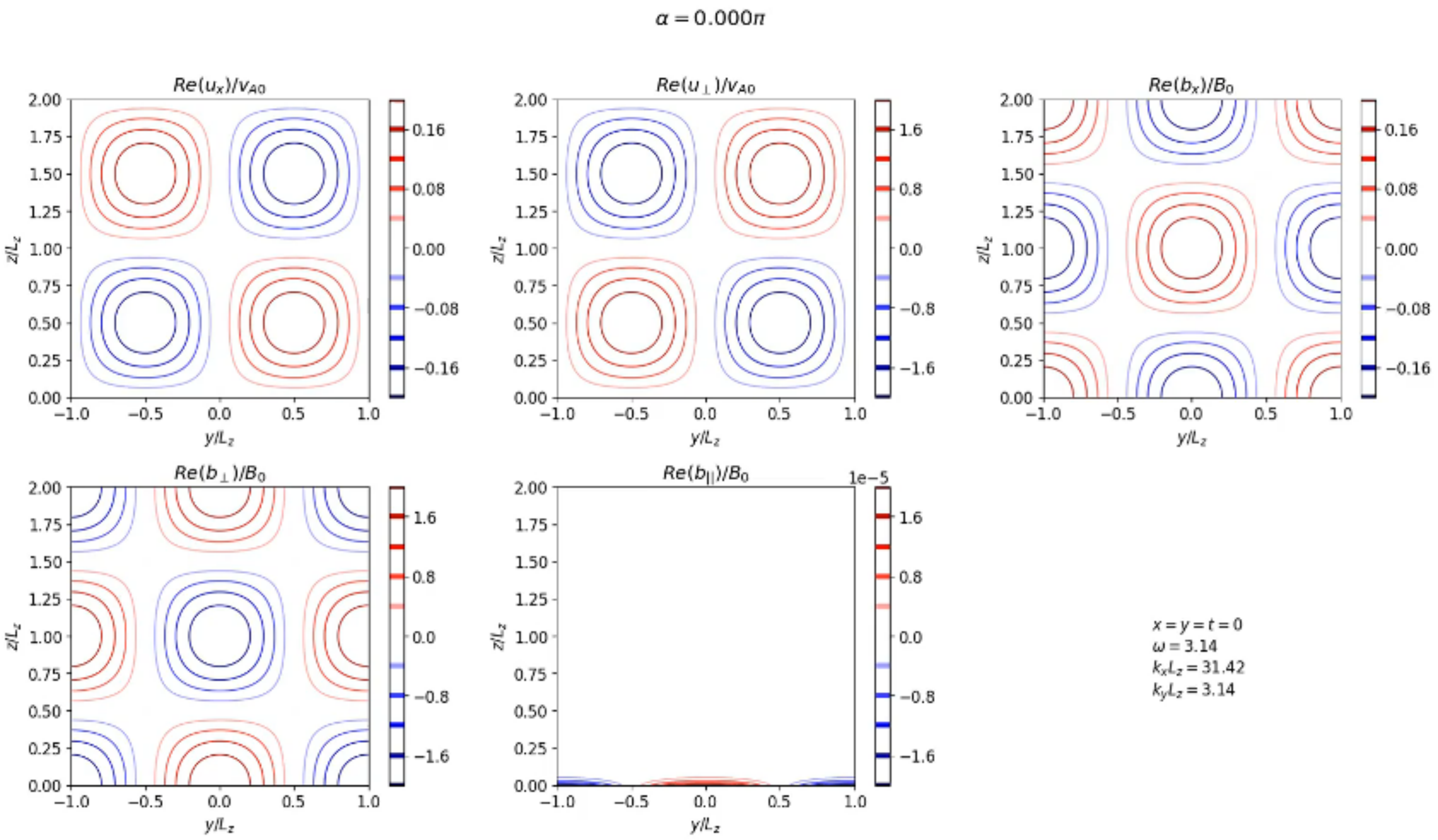
$$b_\parallel(z) = B_0 \frac{I}{v_A} \sin 2\alpha \exp(-k_x z) + O(k_x^{-1})$$

$$b_\perp(z) = 2B_0 \frac{I}{v_A} \left[\underbrace{\cos(k_z^* z) \exp\left(-ik_y \tan\alpha\ z\right)}_{\text{Alfvén wave}} - \underbrace{\sin^2\alpha \exp(-k_x z)}_{\text{Evanescent fast wave}} \right] + O(k_x^{-1})$$

Where: I = incident wave amplitude, $k_z^* = \omega/(v_A cos\alpha)$

Large k_x approximation vs. full solution





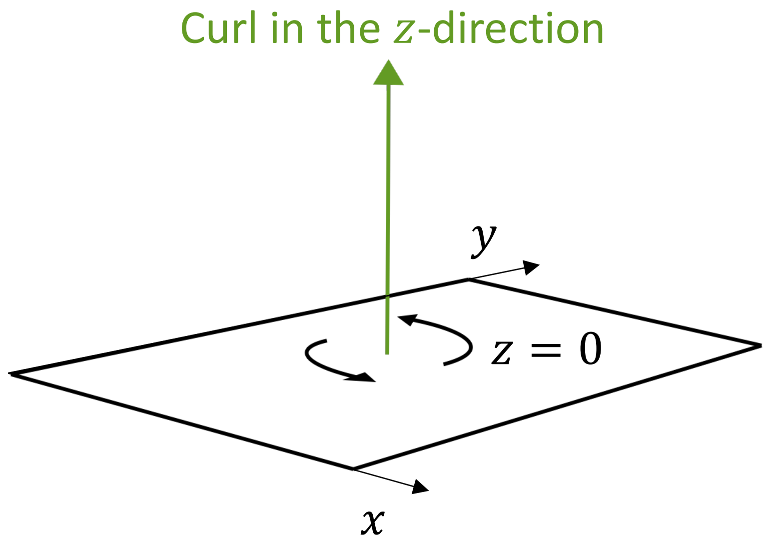
$\alpha = 0.000\pi$
$Re(u_x)/v_{A0}$
$Re(u_\perp)/v_{A0}$
$Re(b_x)/B_0$
$Re(b_\perp)/B_0$
$Re(b_{||})/B_0$
1e−5
z/L_z
y/L_z
$x = y = t = 0$
$\omega = 3.14$
$k_x L_z = 31.42$
$k_y L_z = 3.14$

Why do boundary layers form at $z = 0$?

$$\frac{\partial b_z}{\partial t} = \hat{\boldsymbol{z}} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) = 0$$

$$\Longrightarrow b_z = \cos(\alpha)\, b_\parallel - \sin(\alpha)\, b_\perp = 0$$

$$\Longrightarrow b_\parallel = \tan(\alpha) b_\perp$$



$v_A^2\left(k_x^2 + k_y^2\right) > \omega^2 \Longrightarrow$ Evanescent fast waves

Uniform domain: Summary

- If the field is oblique $\Longrightarrow$ boundary layers form
- Analytic and numerical solutions agree $\Longrightarrow$ leapfrog code is working

Structure

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Normal mode solution

- Assume:

$$u_x, u_\perp, b_x, b_\perp, b_{||} \propto \exp[i(k_y y + k_z z + \omega t)]$$

- Where:

$$k_y \in \mathbb{R} \text{ and } \omega, k_z \in \mathbb{C}$$

$$\omega = \omega_r + i\omega_i$$

$$\frac{\omega_i}{\omega_r} = \epsilon \ll 1$$

Second-order ODE

- This gives:

$$B_0^2 \mathcal{L}^2 \frac{d^2 u_\perp}{dx^2} + B_0^2 \mathcal{L}' \mathcal{L} \frac{d u_\perp}{dx} + [\mathcal{L}^3 + B_0^2(\mathcal{L}^2 \nabla_\perp^2 + \mathcal{L}'' \mathcal{L} - \mathcal{L}'^2))] u_\perp = 0$$

Equation becomes singular if $\mathcal{L} = 0$ ⟵ Wave equation for Alfvén waves

- Where

$$\mathcal{L} = B_0^2 \nabla_\parallel^2 + \mu \rho(x) \omega^2$$

Frobenius solution

- Full solution is given by:

$$u_{\perp}(x) = \underbrace{a_{-1}(x - x_{res})^{-1}}_{\textbf{Singular component}} + \underbrace{\sum_{n=0}^{\infty} a_n (x - x_{res})^n + \ln(x - x_{res}) \sum_{n=1}^{\infty} b_n (x - x_{res})^n}_{\textbf{Regular component}}$$

- Where x_{res} is gives the resonant location

Singular / resonant location (x_{res})

- Given by

$$\mathcal{L} = B_0^2 \nabla_\parallel^2 + \mu\rho(x_{res})\omega^2 = 0$$

- In general $x_{res} \in \mathbb{C}$

- If $\omega \in \mathbb{R}$ and $k_z \in \mathbb{R} \Longrightarrow x_{res} \in \mathbb{R}$

- $\Longrightarrow$ Letting $\omega \in \mathbb{C} \sim$ introducing viscosity

Classifying the solutions for $|x| = O(\epsilon L_z)$

Type I (Singular)	**Type II (Regular)**	**Type III (Laplace)**
$\lvert x_{res} \rvert \leq O(\epsilon\, L_z)$	$\lvert x_{res} \rvert \geq O(L_z)$	$\lvert x_{res} \rvert \geq O(L_z)$
Singular component dominates	Regular component dominates	Regular component dominates
k_z given by $\mathcal{L}(x_{res}) = 0$	$\lvert k_z L_z \rvert \leq O(1)$	$\lvert k_z L_z \rvert \geq O(1/\epsilon)$
$u_\perp(x) \sim \dfrac{1}{x - x_{res}}$	$u_\perp(x) \sim 1 + Cx$	$u_\perp \sim \exp(ik_x x)$

Why Laplace's equation for $|k_z L_z| \geq O(1/\epsilon)$?

- Consider $\rho = \rho_0$
- Solve dispersion relation for k_x

$$\left[\omega^2 + v_A^2 \nabla_\parallel^2\right]\left[\omega^2 - v_A^2\left(k_x^2 + k_y^2 + k_z^2\right)\right] = 0$$

$$\Longrightarrow k_x^2 = \frac{\omega^2}{v_A^2} - k_y^2 - k_z^2$$

$$\Longrightarrow k_x^2 \approx -k_z^2$$

$$\Longrightarrow \frac{\partial^2 u_\perp}{\partial x^2} + \frac{\partial^2 u_\perp}{\partial z^2} = 0$$

Normal mode: Summary

- General solution is given by:

$$u_{\perp} = \exp[i(k_y y + \omega t)] \sum_{n=1}^{\infty} u_{\perp n}(x) \exp(ik_{zn} z)$$

- The terms categorize into:
  - Type I (Singular)
  - Type II (Regular)
  - Type III (Laplace)

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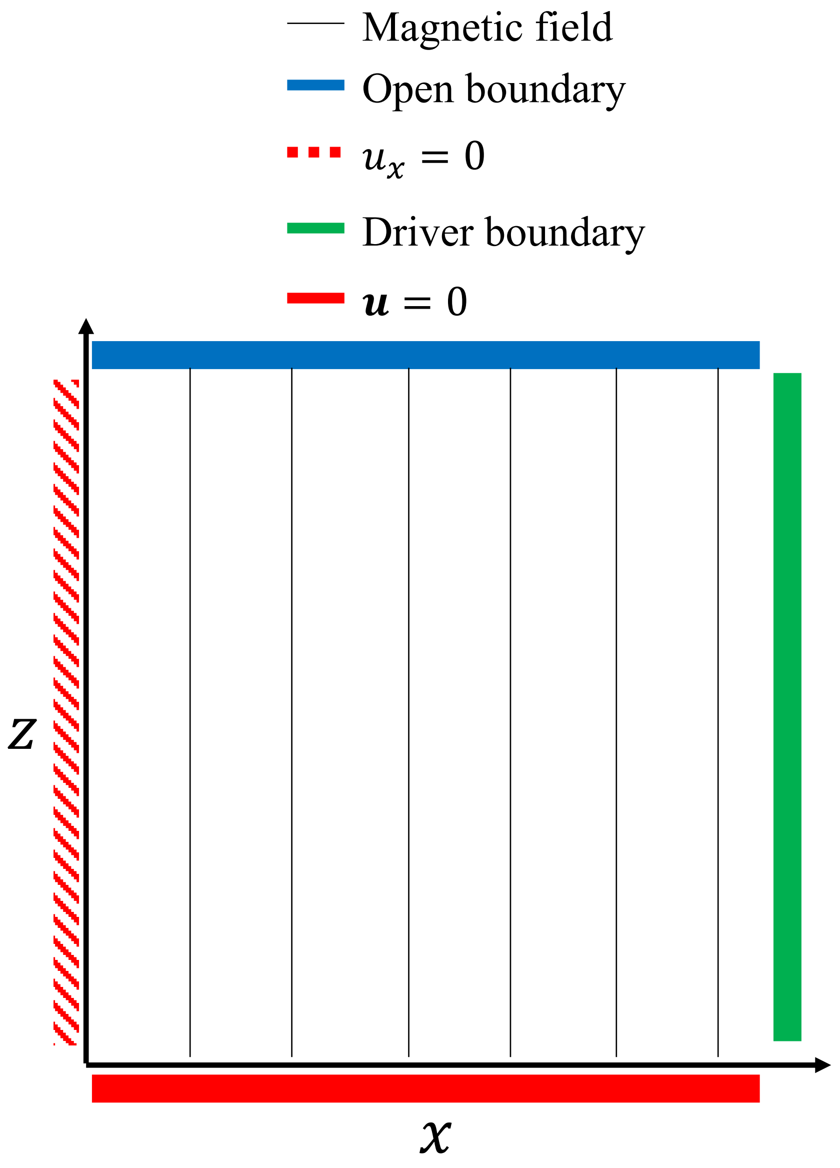
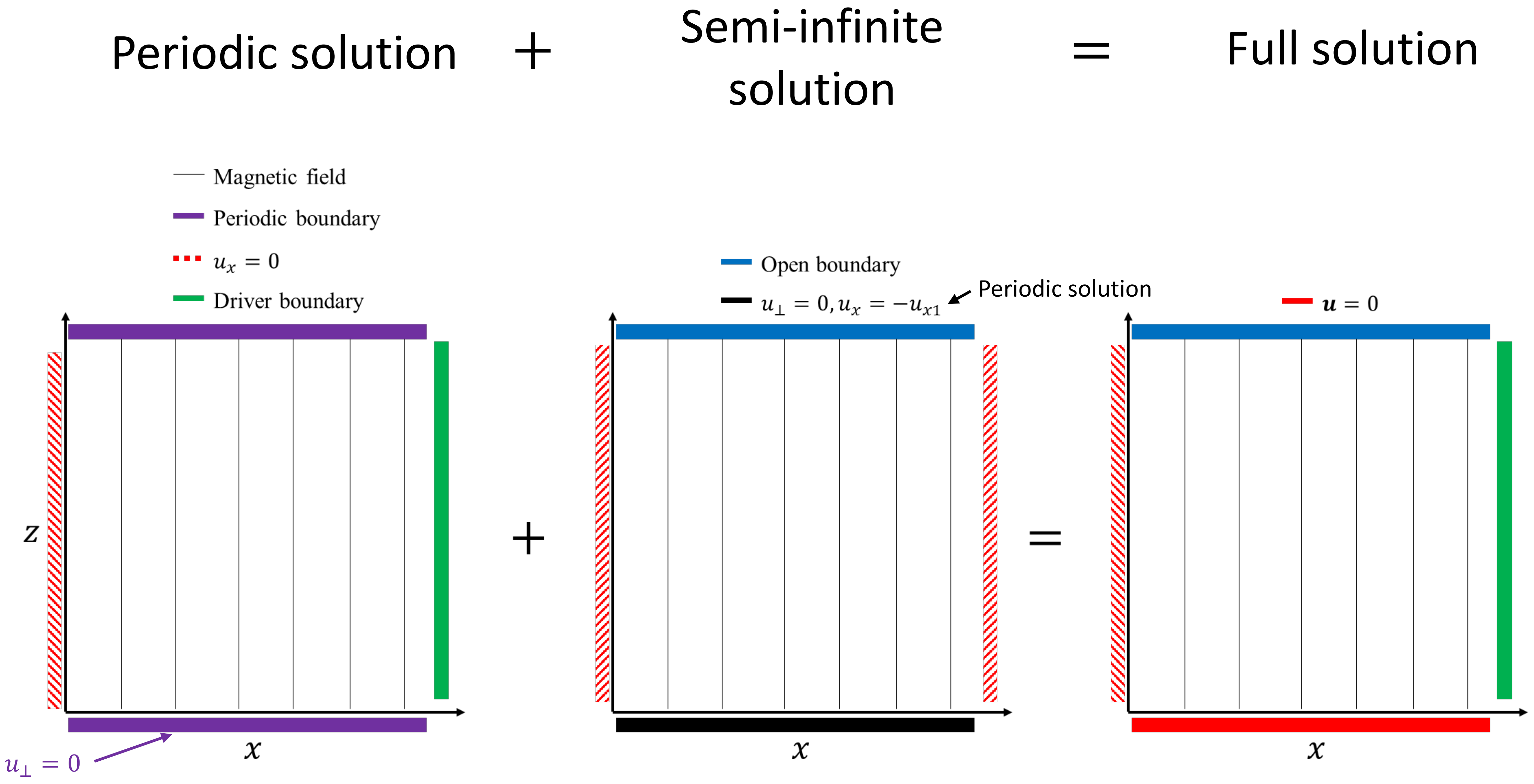
Boundary conditions (BCs)

- $u_x = 0$ at $x = x_{min}$
- $u_x = g(y, z, t)$ at $x = x_{max}$
- $\boldsymbol{u} = 0$ at $z = z_{min}$
- Solutions remain finite $\forall\ z$





Periodic solution
+
Semi-infinite solution
=
Full solution
Magnetic field
Periodic boundary
$u_x = 0$
Driver boundary
z
x
$u_\perp = 0$
+
Open boundary
$u_\perp = 0, u_x = -u_{x1}$
Periodic solution
x
=
$\boldsymbol{u} = 0$
x

Periodic solution

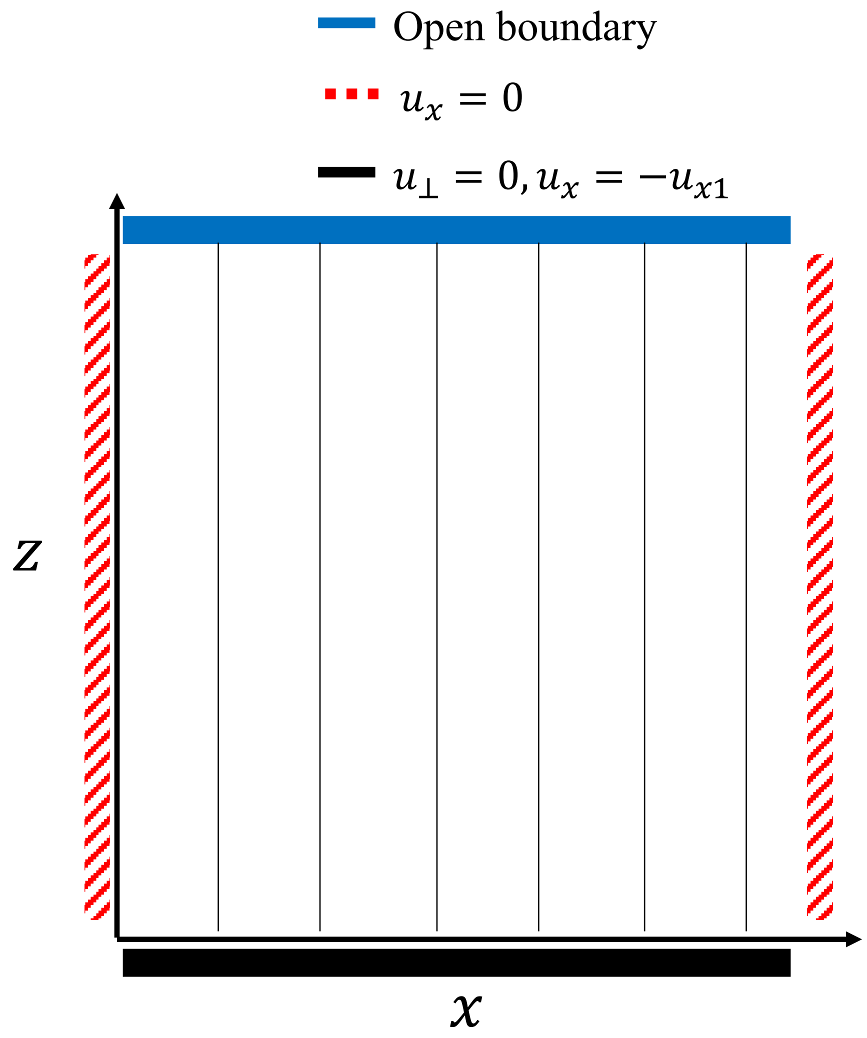
- Driver:

$$g(y,z,t) = \exp[i(k_y y + \omega t)] \sum_{n=-\infty}^{\infty} C_n \exp(ik_{zn} z)$$

- $\Longrightarrow$ Full solution = superposition of normal modes
- If the m'th term satisfies $\mathcal{L}(ix_i) = 0$ then it will dominate
- No BL's

Semi-infinite solution

- $z \in [0, \infty)$
- Solutions remain finite ∀ z
- $u_\perp = 0$
- $u_x = -u_{x1}$ (Periodic solution)



Proposed solution

- Solution is unique

- Note that:
  - For Type III (Laplace) solutions: $u_x = O(u_\perp)$
  - For Type I (singular) solution: $u_x = O(\epsilon \ln \epsilon \, u_\perp)$

- $\Longrightarrow$ Create asymptotic expansion:

$$u_x = w_x(x, y, z, t) + O(\epsilon \ln \epsilon)$$

$$u_\perp = w_\perp(x, y, z, t) + p(x, y, z, t) + O(\epsilon \ln \epsilon)$$

Proposed solution

$w_x(x, 0) = -u_{x1}$ (Periodic solution)

$$u_x = w_x(x, z) + O(\epsilon \ln \epsilon)$$

$$u_\perp = w_\perp(x, z) + p(x, z) + O(\epsilon \ln \epsilon)$$

$p(x, 0) = -w_\perp(x, 0)$

Type III (Laplace solution)

$$\frac{\partial^2 w_x}{\partial x^2} + \frac{\partial^2 w_x}{\partial z^2} = 0$$

- Postulate a solution of the form

$$w_x(x, z) = \sum_{n=-\infty}^{\infty} C_n \exp(ik_n x - |k_n| z)$$

- Satisfying
  - $w_x(x, 0) = -u_{x1}(x, 0)$ (Periodic solution)
  - $w_x(\pm l_x, z) = 0$
  - w_x finite $\forall z$

Type I solution (singular solution)

- Postulate solution of the form

$$p(x,z) = \sum_{j=1}^{\infty} \frac{a_j}{x + b_j} \exp(ik_j z)$$

- Where $a_j, b_j \in \mathbb{C}$
- Satisfying

$$p(x,0) = -w_{\perp}(x,0)$$

- Need to express

$$w_{\perp}(x,0) = -\sum_{j=1}^{\infty} \frac{a_j}{x + b_j}$$

How to write $w_{\perp}(x,0) = \sum_{j=1}^{\infty} \frac{a_j}{x+b_j}$?

$$w_{\perp}(x,0) \approx \frac{P(x)}{Q\,(x)} = \sum_{j=1}^{\infty} \frac{a_j}{x+b_j}$$

Padé approximant

Partial fraction decomposition

- Where:

$$P(x) = \text{Degree } N-1 \text{ polynomial}$$
$$Q(x) = \text{Degree } N \text{ polynomial}$$

What is the Padé approximant?

A portion of the Padé table for the exponential function e^z

$m \backslash n$	0	1	2	3
0	$\frac{1}{1}$	$\frac{1}{1-z}$	$\frac{1}{1-z+\frac{1}{2}z^2}$	$\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$
1	$\frac{1+z}{1}$	$\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$	$\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\frac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$
2	$\frac{1+z+\frac{1}{2}z^2}{1}$	$\frac{1+\frac{2}{3}z+\frac{1}{6}z^2}{1-\frac{1}{3}z}$	$\frac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$	$\frac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$
3	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}{1}$	$\frac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$	$\frac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$	$\frac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$
4	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4}{1}$	$\frac{1+\frac{4}{5}z+\frac{3}{10}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-\frac{1}{5}z}$	$\frac{1+\frac{2}{3}z+\frac{1}{5}z^2+\frac{1}{30}z^3+\frac{1}{360}z^4}{1-\frac{1}{3}z+\frac{1}{30}z^2}$	$\frac{1+\frac{4}{7}z+\frac{1}{7}z^2+\frac{2}{105}z^3+\frac{1}{840}z^4}{1-\frac{3}{7}z+\frac{1}{14}z^2-\frac{1}{210}z^3}$

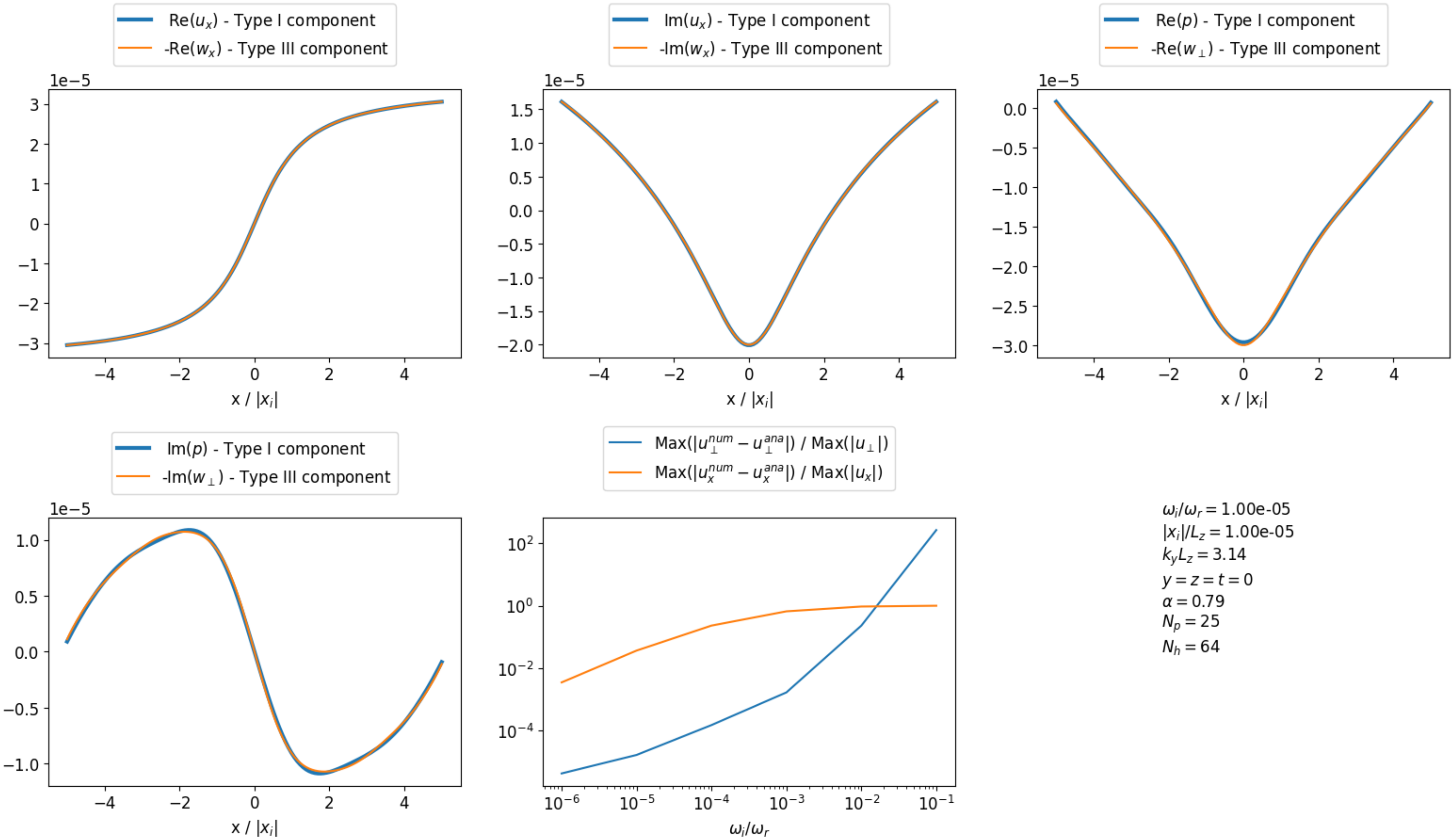
Courtesy: wikipedia.org/Padé_table

Numerical confirmation

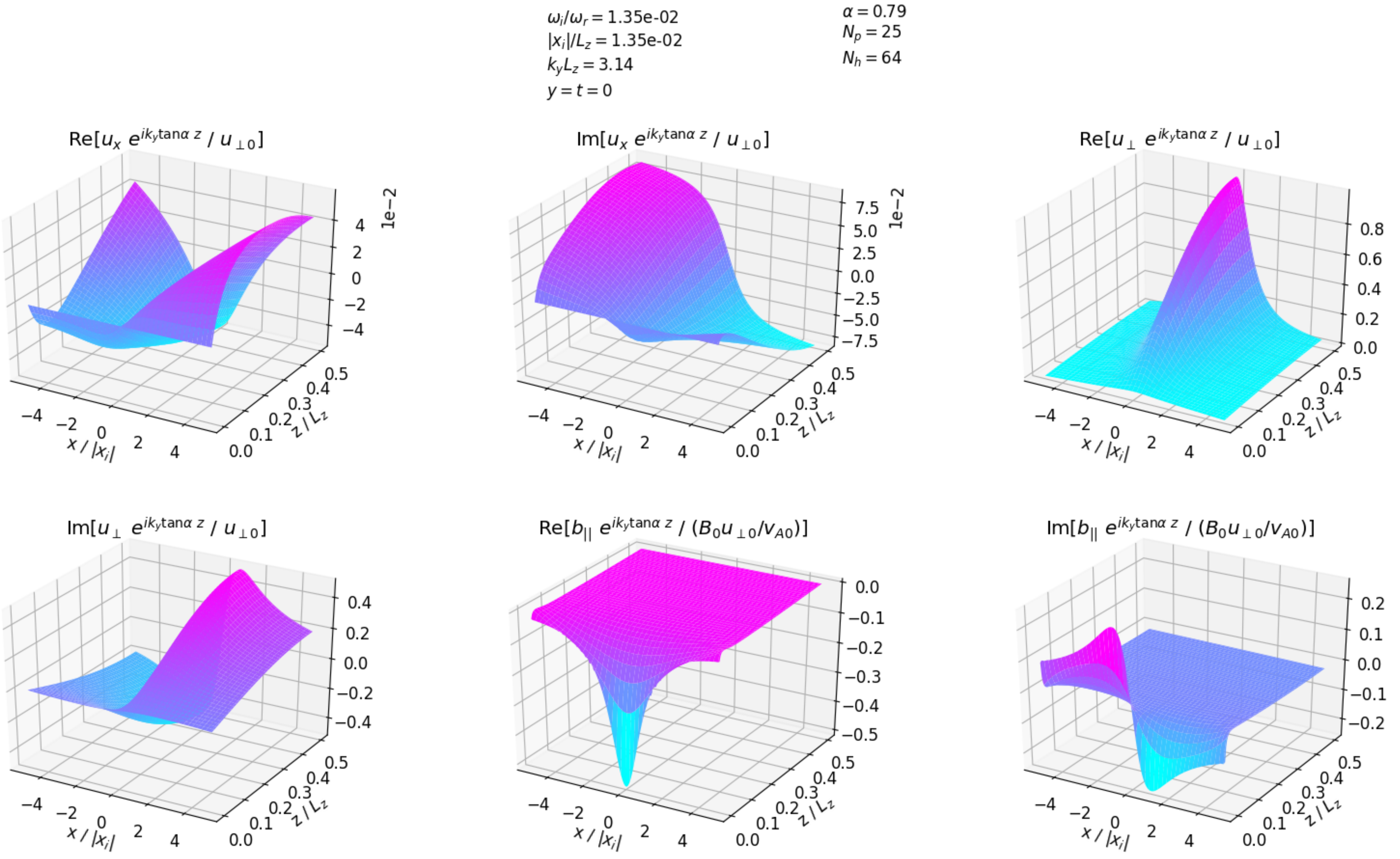
- We choose

$$u_{x1}(x,0) = \ln(x - ix_i)$$

- Where u_{x1} = periodic solution
- We claim we know the z-dependence $\Longrightarrow$ we can calculate corresponding x-dependence numerically
- Check analytic x-dependence and numerical x-dependence agree
- Check BCs are satisfied



Re(u_x) - Type I component
-Re(w_x) - Type III component
Im(u_x) - Type I component
-Im(w_x) - Type III component
Re(p) - Type I component
-Re($w_\perp$) - Type III component
Im(p) - Type I component
-Im($w_\perp$) - Type III component
Max($|u_\perp^{num} - u_\perp^{ana}|$) / Max($|u_\perp|$)
Max($|u_x^{num} - u_x^{ana}|$) / Max($|u_x|$)
1e−5
x / $|x_i|$
ω_i/ω_r
$\omega_i/\omega_r = 1.00$e-05
$|x_i|/L_z = 1.00$e-05
$k_y L_z = 3.14$
$y = z = t = 0$
$\alpha = 0.79$
$N_p = 25$
$N_h = 64$


$\omega_i/\omega_r = 1.35\text{e-}02$
$|x_i|/L_z = 1.35\text{e-}02$
$k_y L_z = 3.14$
$y = t = 0$
$\alpha = 0.79$
$N_p = 25$
$N_h = 64$
$\mathrm{Re}[u_x\ e^{ik_y \tan\alpha\ z}\ /\ u_{\perp 0}]$
$\mathrm{Im}[u_x\ e^{ik_y \tan\alpha\ z}\ /\ u_{\perp 0}]$
$\mathrm{Re}[u_\perp\ e^{ik_y \tan\alpha\ z}\ /\ u_{\perp 0}]$
$\mathrm{Im}[u_\perp\ e^{ik_y \tan\alpha\ z}\ /\ u_{\perp 0}]$
$\mathrm{Re}[b_{||}\ e^{ik_y \tan\alpha\ z}\ /\ (B_0 u_{\perp 0}/v_{A0})]$
$\mathrm{Im}[b_{||}\ e^{ik_y \tan\alpha\ z}\ /\ (B_0 u_{\perp 0}/v_{A0})]$
1e−2
$x\ /\ |x_i|$
$z\ /\ L_z$

Energy of the boundary layer (BL)

- Energy given by:

$$E_{BL} \sim \int_0^\infty b_\parallel^2 \, dz \sim \int_0^\infty k_x \exp(-k_x z)\, dz = 1$$

- $\Longrightarrow E_{BL} \approx$ constant
- BL's are evanescent

$\Longrightarrow$ BL's have a limited effect on the rate of absorption

Summary and conclusions

- Oblique field + solid BCs $\Longrightarrow$ evanescent BLs
- BLs do not stop resonance absorption
- BLs have a limited effect on the rate of absorption

Future work

- Calculate solution for a finite domain
- Confirm results using leapfrog code
- Study in a cylindrical geometry

References

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